Jurnal Matematika Thales (JMT): 2020 ISSN: 2715-1891
Vol. 02 No. 02

(THE GREATEST SOLUTION IN THE INEQUALITY OF $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ WITH $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ ARE A COMPLETE IDEMPOTENT SEMIRINGS OF INTERVAL)

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Abstract. The greatest solution of an inequality $K \otimes X \preceq X \preceq L \odot X$ to solve the optimal control problem for P-Temporal Event Graphs, which is to find the optimal control that meets the constraints on the output and constraints imposed on the adjusted model problem (the model matching problem). We give the greatest solution $K \otimes X \preceq X \preceq L \odot X$ and $X \preceq H$ with K, L, X, H matrices whose are entries in a complete idempotent semirings. Furthermore, the authors examine the existence of a sufficient condition of the projector in the set of solutions of inequality $K \otimes X \preceq X \preceq L \odot X$ with K, L, X matrix whose entries are in the complete idempotent semiring. Projectors can be very necessary to synthesize controllers in manufacturing systems that are constrained by constraints and some industrial applications. The researcher then examines the requirements for the presence of the greatest solution was called projector in the set of solutions of the inequality $\mathbf{K} \otimes \mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L} \odot \mathbf{X}$ with $\mathbf{K}, \mathbf{L}, \mathbf{X}$ matrices whose are entries in an complete idempotent semiring of interval. Researchers describe in more detail the proof of the properties used to resolve the inequality $\mathbf{K} \otimes \mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L} \odot \mathbf{X}$. Before that, we give the greatest solution of the inequality $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ and $\mathbf{X} \preceq \mathbf{G}$ with $\mathbf{K}, \mathbf{L}, \mathbf{X}, \mathbf{G}$ matrices whose are entries in an complete idempotent semiring of interval.

Keywords: Complete idempotent semirings, projector, max plus algebra, complete idempotent semiring of intervals, greatest solution.

2010 Mathematics Subject Classification: 16Y60, 15A80

Abstrak. Solusi terbesar dari pertidaksamaan $K \otimes X \preceq X \preceq L \odot X$ untuk menyelesaikan masalah kontrol optimal untuk P-Temporal Event Graphs, yaitu menemukan kontrol optimal yang memenuhi kendala pada output dan kendala dikenakan pada masalah model yang disesuaikan (model matching problem). Kami memberikan solusi terbaik $K \otimes X \preceq X \preceq L \odot X$ dan $X \preceq H$ dengan matriks K, L, X, H yang merupakan entri dalam semiring idempoten lengkap. Selain itu, penulis memeriksa syarat cukup adanya solusi terbesar yang disebut proyektor dalam himpunan solusi pertidaksamaan $K \otimes X \preceq X \preceq L \odot X$ dengan K, L, X matrix yang entri-entrinya ada di semiring idempoten lengkap. Proyektor dapat sangat diperlukan untuk mensintesis pengontrol dalam sistem manufaktur yang dibatasi oleh kendala dan beberapa aplikasi industri. Peneliti kemudian memeriksa persyaratan untuk adanya proyektor di himpunan solusi dari ketidaksetaraan $K \otimes X \preceq X \preceq L \odot X$ dengan matriks K, L, X yang merupakan entri dalam semiring idempoten lengkap interval. Peneliti menjelaskan lebih rinci bukti sifat yang digunakan untuk menyelesaikan pertidaksamaan. Sebelum itu, kami memberikan solusi terbaik $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ dan $\mathbf{X} \preceq \mathbf{H}$ dengan matriks $\mathbf{K}, \mathbf{L}, \mathbf{X}, \mathbf{H}$ yang entri dalam semiring idempoten lengkap interval.

Kata-kata kunci: semiring idempotent lengkap, proyektor, aljabar maks plus, semiring idempoten lengkap interval, solusi terbesar.

1. INTRODUCTION

When viewed from the structure of algebra, semiring is a generalization of the ring. The idempotent property given to the semiring forms a new structure, which is the idempotent semiring. Many issues in optimization theory and other areas in mathematics are non-linear problems but can be seen as linear over idempotent semiring. The existence of the idempotent property of the sum operation, that is, $\forall p \in S, p \oplus p = p$, can relate a partial order relation to the addition operation, so that the idempotent semirings forms sup - semilattices. An idempotent semirings is closed against an infinite sum and its distribution multiplicity to an infinite sum, so an idempotent semirings is said to be complete. Idempotent semirings generally do not have an inverse of their sum and multiplication operations, except in max plus algebra, so in Baccelli [2] explained that the residuation theory is used in solving equations defined by idempotent semiring. One of the equations discussed is the $K \otimes X \preceq L$ with K, X, L matrices with entries of elements from the complete idempotent semirings of S. The greatest solution of the equation $K \otimes X \preceq L$ can be obtained using residuation theory. Gaubert [8] examines the greatest solution of the $K \otimes X \preceq L$ equation with $K \in S^{n \times p}$ and $X \in S^{p \times 1}$. Furthermore, Hardouin [4] also discusses the greatest solution of the $K \otimes X \preceq L$ equation, but $K \in S^{n \times p}$ and $X \in S^{p \times m}$.

As you are familiar, while idempotent has the multiplicative operation of $\otimes$ distributive to the sum of $\oplus$. Subsequently, Hardouin [4] introduced the duality of the $\otimes$ multiplication operation, which is a dual $\odot$ product that has the distributive $\odot$ distributive property of $\wedge$. In addition to dual products on idempotent semiring, dual products are also given on the top matrix of idempotent semiring notated $K \odot X$, defined by operation $(K \odot X)_{ij} = \bigwedge_{p=1,\ldots,n}(k_{ip} \odot x_{pj})$ with $\wedge$ is the greatest lower bound. Then by finding the dual residuals of the dual products, we can get the smallest solution of the equation $K \odot X \succeq L$. Based on the solution equation $K \otimes X \preceq L$ and $K \odot X \succeq L$, Ouerght and Hardouin [15] explain the greatest solution that satisfies the equation $K \otimes X \preceq X \preceq L \odot X$ with $K, L \in S^{n\times n}$. Ouerght and Hardouin [15] use the greatest solution of this equation to solve optimal control problems for P-temporal event graphs, that is, looking for the greatest control that contented the output and the constraints imposed on the adjusted model problem (the model matching problem). Furthermore, Hardouin together with Brunsch, Maia, Raisch [4] explained the characteristics of the equation $K \otimes X \preceq X \preceq L \odot X$ related to the set of solutions. The author completes the proof of the properties of the equation $K \otimes X \preceq X \preceq L \odot X$.

J.P Quadrat and Cohen [7] have constructed linear projections in idempotent semiring (dioid). One example of idempotent semiring is max plus algebra, so J.P Quadrat and Cohen [7] continue their research on linear projectors in algebra max plus. Projectors can be very necessary to synthesize controllers in manufacturing systems that are constrained by constraints and some industrial applications. As a result, Hardouin together with Brunsch, Maia, Raisch [4] provided a sufficient requirement for projectors in the set of solution equations $K \otimes X \preceq X \preceq L \odot X$.

In the research which did author before, have given Theorem which the greatest solution in the inequality $K \otimes X \preceq X \preceq L \odot X$ with K, L, X are a complete idempotent semirings. Now, we applied in the equality $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ with $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ are semirings idempotent complete of interval and S is a complete idempotent semiring. Furthermore, Hardouin and Brunsch [4] commented that the required situation for the existing of projectors in the set of solution equations $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ with $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ are semirings of interval. The researcher proof different condition with Hardouin[4], the necessary condition for the beingness of projector as the greatest solution in the set of solution equations $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ but with $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ are a complete idempotent semiring of interval and S is an complete idempotent semiring. The author describes in more detail the proof of the properties used to resolve the inequality. But, the the existence of projectors in the set of solution inequality $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ with $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ are a semiring idempotent complete of interval with S is a complete idempotent semiring.

2. MAIN RESULT

2.1. **Semiring Idempotent.** Any arbitrary set which is endowed by two binary operation and hold any axiom, then form a structure algebra. In this article, structure algebra which is explained is an idempotent semiring.

Definition 2.1. *[6] Any arbitrary set of S and two binary operations at S are given, namely the addition of $\oplus$ and the multiplication operation of $\otimes$. The S set with two binary operations $\oplus$ and $\otimes$ is called **semiring idempotent**, denoted $(S, \oplus, \otimes)$, if:*

(i). $(S, \oplus, \varepsilon)$ is an idempotent commutative monoid, i.e. $\forall k \in S(k \oplus k = k)$
(ii). $(S, \otimes, e)$ is monoid
(iii). Operations $\otimes$ are distributive to operations $\oplus$
(iv). There are absorb elements to the operation of $\otimes$, i.e. $\forall k$, $\varepsilon \otimes k = k \otimes \varepsilon = \varepsilon$

An idempotent semiring S could be complete with a canonical order explained as $k \succeq l$ if and only if $k = k \oplus l$. Thus, S is **sup-semilattice**, and $k \oplus l$ is the lower upper bound of k and l. A semiring of S is said **complete semiring** if

(i). $\forall k_i \in S, i \in I, \bigoplus_{i \in I} k_i \in S$ item $\forall k_i, l \in S, i \in I, (\bigoplus_{i \in I} k_i) \otimes l = \bigoplus_{i \in I}(k_i \otimes l)$

Furthermore, the sum of all elements in the complete semiring of S is denoted $\top = \bigoplus_{x \in S} x$. In **complete lattice** defined $k \succeq l \Leftrightarrow k = k \wedge l$. The complete slope of S has the same structure as **complete lattice** which has the greatest lower bound of k and l denoted $k \wedge l$

2.2. **Theory of Residuation.** The theory of residuation explains the inverse of the mapping which preserves the relation $\preceq$ sequence which is defined on the ordered set, that is, the set which is supplemented by a partial order relation. In the following, we will give definitions of residuated mapping and dually residuated mapping which are widely used in proving some of the following theorems.

Definition 2.2. *[2] Given the ordered set $\mathcal{R}$ and $\mathcal{S}$ and the canonical relation $\preceq$. The mapping that preserves the order $f : \mathcal{R} \to \mathcal{S}$ is called to be a **mapping residuated** if for every $y \in \mathcal{S}$, the lower upper bound of the subset of $\{x \in \mathcal{R} \mid f(x) \preceq y\}$ exists and is a element of the subset, the element is indicated $f^\sharp(y)$. Meanwhile, mapping $f^\sharp : \mathcal{S} \to \mathcal{R}$ is said to be a **residual** from f. If f is a residuated mapping, then $f^\sharp$ is an unique mapping that preserves the sequence such that*

$$f \circ f^\sharp \preceq Id_{\mathcal{S}} \; dan \; f^\sharp \circ f \succeq Id_{\mathcal{R}} \tag{2.1}$$

where Id represents the identity mapping of $\mathcal{R}$ and $\mathcal{S}$.
*The g mapping is said to be a **mapping dually residuated** if for every $y \in \mathcal{S}$, the greatest lower bound of the $\{x \in \mathcal{R} \mid g(x) \succeq y\}$ exists and is a element of the subset, this element is denoted $g^b(y)$. Next, mapping $g^b : \mathcal{S} \to \mathcal{R}$ is said to be a **dual residual** from g. While g is **mapping dually residuated**, g^b is a unique mapping that preserves the sequence such that*

$$g \circ g^b \preceq Id_{\mathcal{S}} \; and \; g^b \circ g \succeq Id_{\mathcal{R}} \tag{2.2}$$

Theorem 2.3. *[2] Given idempotent semirings $\mathcal{R}$ and $\mathcal{Q}$. If given $h : \mathcal{R} \to \mathcal{Q}$ and $f : \mathcal{R} \to \mathcal{Q}$ are a residuated mapping then hold the next property:*

$$f \preceq h \Leftrightarrow h^\sharp \preceq f^\sharp$$

Theorem 2.4. *[2] Given idempotent semirings $\mathcal{R}$ and $\mathcal{Q}$. If we have $h : \mathcal{R} \to \mathcal{Q}$ and $g : \mathcal{R} \to \mathcal{Q}$ are dually residuated mappings then hold the following proposotion:*

$$g \preceq h \Leftrightarrow h^b \preceq g^b$$

Proposition 2.5. *[2] Given a semiring $\mathcal{S}$ and $\mathcal{Q}$. Given $f : \mathcal{S} \to \mathcal{Q}$ is a residuated mapping, a mapping $P_f = f \circ f^\sharp$ is projector and $P_f(c)$ with $c \in \mathcal{Q}$ is the greatest element in $Im\, f$ less than or equal c. Given $g : \mathcal{S} \to \mathcal{Q}$ is a dually residuated mapping, the mapping $P_g = g \circ g^b$ is the projector and $P_g(d)$ with $d \in \mathcal{Q}$ is the smallest element in $Im\, g$ greater than or equal d.*

Proposition 2.6. [**?**]*Given $f : \mathcal{D} \to \mathcal{E}$ is a residuated mapping, $g : \mathcal{D} \to \mathcal{E}$ is a dual residuated mapping and $\mathcal{D}_{sub}$ ($\mathcal{E}_{sub}$) is a complete subsemiring of $\mathcal{D}$ ($\mathcal{E}$)*

(1) the mapping $f_{|\mathcal{D}_{sub}}$ is a residuated mapping, and its residual is given as the follows :

$$(f_{|\mathcal{D}_{sub}})^\sharp = (f \circ Id_{|\mathcal{D}_{sub}})^\sharp = (Id_{|\mathcal{D}_{sub}})^\sharp \circ f^\sharp$$

(2) If $Im\, f \subset \mathcal{E}_{sub}$ then mapping $\mathcal{E}_{sub}|f$ is residuated mapping and its residual is given as the follows :

$$(\mathcal{E}_{sub}|f)^\sharp = f^\sharp \circ Id_{|\mathcal{E}_{sub}} = (f^\sharp)_{\mathcal{E}_{sub}}$$

(3) The mapping $g_{|\mathcal{D}_{sub}}$ is dual residuated mapping and its dual residual is set as the follows :

$$(g_{|\mathcal{D}_{sub}})^\flat = (g \circ Id_{|\mathcal{D}_{sub}})^\flat = (Id_{|\mathcal{D}_{sub}})^\flat \circ g^\flat$$

(4) if $Im\, g \subset \mathcal{E}_{sub}$ then mapping $\mathcal{E}_{sub}|g$ is dual residuated mapping and its dual residual is set as the follows :

$$(\mathcal{E}_{sub}|g)^\flat = g^\flat \circ Id_{|\mathcal{E}_{sub}} = (g^\flat)_{\mathcal{E}_{sub}}$$

Theorem 2.7. [**?**]*Given a semiring $\mathcal{S}$. If a mapping $h : \mathcal{S} \to \mathcal{S}$ is residuated mapping and $g : \mathcal{S} \to \mathcal{S}$ is a dually residuated mapping, then hold the next proposition:*

$$\begin{aligned} &h\, is\, a\, closure\, mapping \Leftrightarrow h^\sharp\, is\, s\, dual\, closure\, mapping \\ &\Leftrightarrow h^\sharp \circ h = h \Leftrightarrow h \circ h^\sharp = h^\sharp \end{aligned} \tag{2.3}$$

$$\begin{aligned} &g\, is\, a\, dual\, closure\, mapping \Leftrightarrow g^b\, is\, a\, closure\, mapping \\ &\Leftrightarrow g \circ g^b = g \Leftrightarrow g^b \circ g = g^b \end{aligned} \tag{2.4}$$

Proposition 2.8. [**?**]*Given S is a semiring, $h : S \to S$, $g : S \to S$, and $f : S \to S$ are three mappings, and asumption g and f are two closure mappings which is residuated. The statement is equavalent :*

$$Im\, h \subset Im\, f \Leftrightarrow f \circ h = h$$
$$g \preceq f \Leftrightarrow f \circ g = f = g^\sharp \circ f \Leftrightarrow Im\, f \subset Im\, g \Leftrightarrow Im\, f \subset Im\, g^\sharp$$

Proposition 2.9. *Given S a complete idempotent semirings. For all $k, l, o \in S$.*

(1) $(k\backslash l) \wedge (o\backslash l) = (k \oplus o)\backslash l$
(2) $(k \otimes l)\backslash x = l\backslash(k\backslash x)$

Definition 2.10. ***Kleene Star*** [**?**]*Given S a complete semiring. A aditive closure of matrix $K \in S^{n\times n}$ is explained as following :*

$$KS : S^{n\times n} \to S^{n\times n}, K \mapsto K^* = \bigoplus_{i\in\mathbb{N}_0} K^i$$

where $K^0 = E, K^n = K \otimes K^{n-1}$ and E a identity matrix, i.e. $\forall i, j \in [1,n], E_{ii} = e$ and $E_{ij} = \varepsilon$ if $i \neq j$.

Proposition 2.11. [?] *Given $K \in S^{n\times n}$ and $X \in S^{n\times p}$. According Definition 2.10, Mappings $\mathfrak{L}_{K^*} : S^{n\times p} \to S^{n\times p}, X \mapsto K^* \otimes X$ are closure mappings. Therefore,*

$$K^* \otimes K^\star \otimes X = K^* \otimes X$$

As consequent, that is equivalent with

$$X = K^* \otimes X \Leftrightarrow X \in Im_{\mathfrak{L}_{K^*}}$$

A mapping $\mathfrak{L}^\sharp_{K^*}$ is a dually closure mapping. Therefore,

$$K^* \setminus K^* \setminus X = K^* \setminus X \tag{2.5}$$

According an Equation 2.3,$\mathfrak{L}_{K^*} \circ \mathfrak{L}^\sharp_{K^*} = \mathfrak{L}^\sharp_{K^*}$ dan $\mathfrak{L}^\sharp_{K^*} \circ \mathfrak{L}_{K^*} = \mathfrak{L}_{K^*}$. Therefore,

$$K^* \otimes (K^* \setminus X) = K^* \setminus X \tag{2.6}$$

and

$$K^* \setminus (K^* \otimes X) = K^* \otimes X \tag{2.7}$$

According Proposition 2.5, The Equation 3.2 is meant $\mathfrak{L}^\sharp_{K^*}$, which is projector on $Im\mathfrak{L}_{K^*}$. Given $L \in S^{n\times n}$ such that $L^* \preceq K^*$ i.e. $\mathfrak{L}_{L^*} \preceq \mathfrak{L}_{K^*}$ then according Proposition 2.8, the following statement is equivalent :

$$L^* \preceq K^* \Leftrightarrow K^*L^*X = K^*X = L^* \setminus (K^*X) \Leftrightarrow Im\,\mathfrak{L}_{K^*} \subset Im\,\mathfrak{L}_{L^*} \Leftrightarrow Im\,\mathfrak{L}_{K^*} \subset Im\,\mathfrak{L}^\sharp_{L^*} \tag{2.8}$$

Proposition 2.12. *Given S a complete idempotent semiring. A matrix $K \in S^{n\times n}$ dan $X \in S^{n\times p}$. A mapping $\mathfrak{L}_{A^*} : S^{n\times p} \to S^{n\times p}$, $X \mapsto K^* \otimes X$ is a closure mapping, therefore,*

$$K^* \otimes K^* \otimes X = K^* \otimes X$$

As a consequent, the following statement is equivalent :

$$X = K^* \otimes X \Leftrightarrow X \in Im\ \mathfrak{L}_{K^*} \tag{2.9}$$

Lemma 2.13. *Given a complete idempotent semiring S and a matrix $K \in S^{n\times n}$ dan $X \in S^{n\times p}$. The next statement is equivalent :*

(1) $X \preceq K \backslash X$
(2) $X \succeq K \otimes X$
(3) $X = K^* \otimes X$
(4) $X = K^* \backslash X$

Lemma 2.14. [?] *Given $K \in S^{n\times n}$ and $X \in S^{n\times p}$. The next statement is equvalent :*

$$X \preceq K \setminus X \Leftrightarrow X \succeq KX \Leftrightarrow X = K^*X \Leftrightarrow X = K^* \setminus X$$

3. DUAL PRODUCT OVER A COMPLETE IDEMPONTENT SEMIRINGS

In the Thomas Brunch article[4], dual products are defined on the side. However, complete and idempotent conditions are required. Full conditions are needed because there is a top element which is the definition of an infinite number of elements. The idempotent requirement is needed because of the assumption that the distributive $\wedge$ of an infinite element. So $\wedge$ needs two elements to have a partial order relation. Even though a semiring can be equipped with partial order relations if there is an idempotent propperty. Therefore, dual products are defined in a complete idempotent semiring

Definition 3.1. *(**(Dual product)**) Given a complete idempotent semiring S, dual products in S, denoted $\odot$, are rules that are assumed and must have e as a neutral element, ie $(S, e, \odot)$ monoid. Furthermore, this dual product is supposed to be distributive of $\wedge$ from infinite elements, and the element $\top$ has an absorbing property that is $\top \odot k = k \odot \top = \top$.*

Definition 3.2. *(**(Dual matrix product)**) Given a complete idempotent semiring S and $K \in S^{n\times p}, L \in S^{p\times m}$ and $O \in S^{n\times m}$ is a matrix, then $O = K \odot L$ is explained as next:*

$$O_{ij} = (K \odot L)_{ij} = \bigwedge_{k=1\ldots p} (k_{im} \odot l_{mj})$$

The identiy matrix is denoted $E^{\odot}$ such that the entries are $E^{\odot}_{ii} = e$ and $E^{\odot}_{ij} = \top$ for $i \neq j$.

In the Brunch Thomas article [4], S is just a semiring. it is lacking if it is a sufficient condition because $\wedge$ requires an idempotent S trait. This idempotent property is needed so that each element in S could be equipped with a partial order relation. so $\wedge$ is an upper semicontinous mapping.

Proposition 3.3. *Given a idempotent semiring S and $K \in S^{p\times n}, X \in S^{n\times m}$ is a matrix, mapping $\wedge_K : S^{n\times m} \to S^{p\times m}, X \mapsto K \odot X$ is an upper semicontinuous mapping, i.e.*

$$\wedge_K(\bigwedge_{X\in\mathcal{X}} X) = \bigwedge_{X\in\mathcal{X}} \wedge_K(X)$$

In the Thomas brunch article [4], it was only given S a semiring. Even though $\wedge_K : S^{p\times m} \mapsto S^{n\times m}$ is dually residuated. As such, elements in S must be able to order need idempotent properties in the semiring as S. Therefore, S must be an idempotent semiring.

Corollary 3.4. *Given a complete idempotent semiring S and $K \in S^{n\times p}$ matrix. Mapping $\wedge_K : S^{p\times m} \to S^{n\times m}, X \mapsto K \odot X$ is dual residuation, and dual residuals are denoted*

$$\wedge_K^{\flat} : S^{n\times m} \to S^{p\times m}, X \mapsto K \bullet X$$

with rules:

$$(K \bullet X)_{ij} = \bigoplus_{q=1}^{n} k_{qi} \bullet x_{qj}$$

and $\top \bullet x = \varepsilon, \varepsilon \bullet x = \top, \varepsilon \bullet \varepsilon = \varepsilon$

On the thomas brunch [4] article, is endowed on a semiring. However, that is not enough because $\wedge$ needs a partial order relation to determine the lower bound of $\wedge$. So $\bigwedge$ must be endowed on an idempotent semiring.

Definition 3.5. *Given S is a idempotent semiring. $\bigwedge$ closure of $L \in S^{n\times n}$ is defined as*

$$L_* = \bigwedge_{q\in\mathbb{N}_0} L^{\odot q}$$

where $L^{\odot 0} = E^{\odot}$ and $L^{\odot k} = L \odot L^{\odot(q-1)}$

In the Thomas brunch article [4], the proposition below is given only S while it is a complete semiring. this is not enough, because the proof is needed $\wedge_L^\flat$ is lower semi-continuous. The mapping $\wedge_L^b : S^{n\times m} \mapsto S^{p\times m}$ is the lower semi-continuous if S is a complete ordered set. A complete order set can be formed if it is provided with a partial order relation. For a set a partial order relation can be endowed if the elementin set are idempotent. So set S for Proposition 3.6 must be a complete idempotent semiring.

Proposition 3.6. *Given S a complete idempotent semirings. A matrix $K \in S^{n\times p}$, $L \in S^{n\times r}$, and $X \in S^{p\times q}$. If every entry $k_{ij} \in K$ hold $b_{ij} \bullet (k \otimes x) = (l_{ij} \bullet k) \otimes x$ for all $k, x \in S$, then*

$$L \bullet (K \otimes X) = (L \bullet K) \otimes X$$

In the Thomas Brunch article [4], semiring S is not given as idempotent. A mapping $\wedge_{L_*}$ is a mapping of $X \mapsto L_* \odot X$. Whereas L_* is defined as $\bigwedge_{q\in\mathbb{N}_0} L^{\odot q}$ so it requires a partial order relation. As such, S must be an idempotent semiring.

Proposition 3.7. *Given S is an idempotent semiring, $L \in \mathcal{S}^{n\times n}$ and $X \in \mathcal{S}^{n\times p}$. Because Λ_L is upper semicontinous and is based on the Definition 3.5, the mapping $\Lambda_{L_*} : \mathcal{S}^{n\times p} \mapsto \mathcal{S}^{n\times p}$, $X \mapsto L_* \odot X$ is a dual closure mapping, therefore,*

$$L_* \odot L_* \odot X = L_* \odot X \tag{3.1}$$

and consequently, fulfill the following equivalents

$$X = L_* \odot X \Leftrightarrow X \in Im\Lambda_{L_*} \tag{3.2}$$

In the Thomas Brunch article [4], S is not described as idempotent. Note that L_* is defined as $\bigwedge_{q\in\mathbb{N}_0} L^{\odot q}$ and is described for proof of $2 \Rightarrow 3$ and $4 \Rightarrow 1$ so that you need partial order relations. As such, S must be idempotent semiring.

Proposition 3.8. *Given S is a complete idempotent semiring and $L \in S^{n\times n}$ dan $X \in S^{n\times p}$ are matrix. The statement below is equivalent:*

(1) $X \preceq L \odot X$

(2) $L \bullet X \preceq X$
(3) $L_* \bullet X = X$
(4) $L_* \odot X = X$

In the Thomas Brunch article [4], S is given a semiring. Note the mapping $\mathfrak{L}_{K_*}$ is defined as a mapping of $X \mapsto K^* \otimes X$ so it requires S is a complete semiring so that K^* is defined. In addition to the mapping $\wedge_{L_*}$, a defined $X \mapsto L_* \odot X$ requires a S idempotent condition so that L_* is defined, so S must be a complete idempotent semiring.

Proposition 3.9. *Given a complete idempotent semiring S dan $K, L \in S^{n\times n}$ dan $X \in S^{n\times m}$. The next statement is equivalent :*

$$K \otimes X \preceq X \preceq L \odot X \Leftrightarrow X \in Im\ \mathfrak{L}_{K^*} \cap Im\ \Lambda_{L_*}$$

4. CONCLUDING REMARK5

4.1. **Projector in The Solution Set of An Inequality** $K \otimes X \preceq X \preceq L \odot X$**.** This Theorem motivated from Theorem in Ouergh, et all(2006)[15]. That theorem is explained as follows :

Proposition 4.1. [**?**]*Let us assumed a dioid $\mathfrak{D}$, a reticulated group $\mathfrak{G} \subset \mathfrak{D}$ and matrices $A, G \in \mathfrak{D}^{n\times n}$ and each entry of G in $\mathfrak{G}$. The greatest solution of X such that :*

$$A \otimes X \preceq X \preceq G \odot X \text{ and } X \preceq B$$

is

$$\widehat{X} = ((G_* \bullet A^*)^*)\backslash B$$

In article which is written by the Ouergh, it has not yet been defined as a dual product operation, so Proposition 4.1 requires element of G in the reticulated group. However, this research has defined dual product operations so that these requirements are not needed. Then the following theorem can be given

Proposition 4.2. *Given a complete idempotent semiring S, and matrix $K, L, G \in S^{n\times n}$. The greatest solution X which hold :*

$$K \otimes X \preceq X \preceq L \odot X \text{ and } X \preceq G$$

is

$$\widehat{X} = ((L_* \bullet K^*)^*)\backslash G$$

Proof. (1) We are shown $K \otimes X \preceq X \preceq L \odot X$ dan $X \preceq G \Rightarrow X \preceq \widehat{X}$. According Proposition 3.9, $K \otimes X \preceq X \preceq L \odot X \Leftrightarrow X \in Im\ \mathfrak{L}_{K^*} \cap Im\ \Lambda_{L_*}$. It means X must hold $X = L_* \bullet (K^* \otimes X)$.

$$\begin{aligned} X = L_* \bullet (K^* \otimes X) \quad &\Leftrightarrow \quad X = (L_* \bullet K^*) \otimes X \ (\textit{Proposisi } 3.6) \\ &\Leftrightarrow \quad X = ((L_* \bullet K^*)^*)\backslash X \ (\textit{Lemma } 2.13) \end{aligned}$$

Futhermore, $K \otimes X \preceq X \preceq L \odot X$ and $X \preceq G \Leftrightarrow X = ((L_* \bullet K^*)^*)\backslash X$ and $X \preceq G$. According Theorem 2.13 $X = ((L_* \bullet K^*)^*)\backslash X \Leftrightarrow (L_* \bullet K^*) \otimes X \preceq X$. Because $(L_* \bullet K^*) \otimes X \preceq X$ dan $X \preceq G$ then $X \preceq \widehat{X} = ((L_* \bullet K^*)^*)\backslash G$.

(2) We will be shown $\widehat{X} \preceq G$, $\widehat{X} = K^* \otimes \widehat{X}$, and $\widehat{X} = L_* \odot \widehat{X}$.

First we will be shown $\widehat{X} \in Im\ \mathfrak{L}_{K^*}$ which equivalent $\widehat{X} = K^* \otimes \widehat{X} = K^* \backslash \widehat{X}$. According Lemma 2.13, $\widehat{X}$ hold

$$(L_* \bullet K^*) \otimes \widehat{X} \preceq \widehat{X} \preceq (L_* \bullet K^*) \backslash \widehat{X} \tag{4.1}$$

Because $\mathfrak{L}_{K^*}$ an isoton mapping and $\widehat{X} \preceq (L_* \bullet K^*) \backslash \widehat{X}$, then $K^* \otimes \widehat{X} \preceq K^* \otimes ((L_* \bullet K^*) \backslash \widehat{X})$. According Theorem 2.3 is found

$$K^* \backslash \widehat{X} \succeq K^* \backslash ((L_* \bullet K^*) \backslash \widehat{X})$$

Furthermore,

$$\begin{aligned} K^* \backslash ((L_* \bullet K^*) \backslash \widehat{X}) &= ((L_* \bullet K^*) \otimes K^*) \backslash \widehat{X} (Proposition\ 2.9) \\ &= (L_* \bullet (K^* \otimes K^*)) \backslash \widehat{X} (Proposition\ 3.6) \\ &= (L_* \bullet K^*) \backslash \widehat{X} \end{aligned}$$

So we are found $K^* \backslash \widehat{X} \succeq (L_* \bullet K^*) \backslash \widehat{X}$. According an inequality 4.1, $(L_* \bullet K^*) \backslash \widehat{X} \succeq \widehat{X}$. As a result, $K^* \backslash \widehat{X} \succeq (L_* \bullet K^*) \backslash \widehat{X} \succeq \widehat{X}$. So we are found $K^* \backslash \widehat{X} \succeq \widehat{X}$. So that, $\widehat{X} \succeq K^* \backslash \widehat{X}$ (because $K^* \succeq E$) then $K^* \backslash \widehat{X} = \widehat{X}$, i.e. $\widehat{X} \in Im\ \mathfrak{L}^*$.

The second step, we are shown $\widehat{X} \in Im\ \Lambda_{L_*}$, i.e. $\widehat{X} = L_* \odot \widehat{X} = L_* \bullet \widehat{X}$. From an inequality 4.1 is found $\widehat{X} \succeq (L_* \bullet K^*) \otimes \widehat{X} = L_* \bullet (K^* \otimes \widehat{X}) = L_* \bullet \widehat{X}$ (because $\widehat{X} = K^* \otimes \widehat{X}$). in the other side, $L_* \preceq E^{\odot}$. On account of Λ_{L_*} is an isoton mapping, then $L_* \odot \widehat{X} \preceq E^{\odot} \odot \widehat{X}$. Therefore, according Theorem 2.4 we are found $\widehat{X} \preceq L_* \bullet \widehat{X}$. So that $\widehat{X} = L_* \bullet \widehat{X} = L_* \odot \widehat{X}$.

The last step because $(L_* \bullet K^*)^* \succeq E$, then $(L_* \bullet K^*)^* \odot G \succeq E \odot G$. According Theorem 2.3, we are found $(L_* \bullet K^*)^* \backslash G \preceq G$. So $\widehat{X} \preceq G$

□

In the Thomas Brunch article [?], S is given in one piece. Note the $\mathfrak{P}$ is defined $\mathfrak{P} : S^{n \times m} \to S^{n \times m}, X \mapsto (L_* \bullet K^*)^* \setminus X$. Because K^* can be defined, it requires a complete S condition while L_* needs a pasial relation so S must be idempotent semiring. Thus, S must be a complete idempotent semiring.

Proposition 4.3. *Given S a complete idempotent semiring and $K, L \in S^{n \times n}$ and $X \in S^{n \times m}$. If $\forall X$, an equality $L_* \bullet (K^* \otimes X) = (L_* \bullet K^*) \otimes X$ is hold, so a mapping*

$$\mathfrak{P} : S^{n \times m} \to S^{n \times m}, X \mapsto (L_* \bullet K^*)^* \setminus X \tag{4.2}$$

is projector in $Im\mathfrak{L}_{K^*} \cap Im\Lambda_{L_*}$. Officially,

$$\mathfrak{P}(X) = \{\bigvee Y | Y \preceq X\, and\, Y \in Im\mathfrak{L}_{K^*} \cap Im\Lambda_{L_*}\} \tag{4.3}$$

Proof. According an Equation 3.1 and an Equation 3.2, $\mathfrak{P}$ is projector on an image $\mathfrak{L}_{(L_* \bullet K^*)^*}$ and $\mathfrak{P}(X) \preceq X$. As claimed by Definition 3.5, $L_* \preceq E^{\odot}$, then $L_* \bullet K^* \succeq E^{\odot} \bullet K^* = K^*$ and $(L_* \bullet K^*)^* \succeq (K^*)^*$, which is according Equation 2.8, so that $Im\, \mathfrak{L}_{(L_* \bullet K^*)^*} \subset Im\, \mathfrak{L}_{K^*}$. therefore, $\mathfrak{P}(X) \in Im\, \mathfrak{L}_{K^*}$.
Because $\mathfrak{P}(X) \in Im\, \mathfrak{L}_{(L_* \bullet K^*)^*}$ means $\mathfrak{P}(X) = (L_* \bullet K^*) * \mathfrak{P}(X)$. According Lemma

2.14,that equation equivalent with $\mathfrak{P}(X) \succeq (L_* \bullet K^*) \otimes \mathfrak{P}(X)$. Bacause of the assumtion, is found an equation $(L_* \bullet K^*) \otimes \mathfrak{P}(X) = L_*(K^* \otimes \mathfrak{P}(X))$. Furthermore, $\mathfrak{P}(X) \in Im\,\mathfrak{L}_{K^*}$, so that $K^* \otimes \mathfrak{P}(X) = \mathfrak{P}(X)$. Therefore,

$$\mathfrak{P}(X) \succeq (L_* \bullet K^*) \otimes \mathfrak{P}(X) = L_* \bullet (K^* \otimes \mathfrak{P}(X)) = L_* \bullet \mathfrak{P}(X)$$

in the other side, $L_* \preceq E^{\odot}$ then

$$L_* \bullet \mathfrak{P}(X) \succeq E^{\odot} \bullet \mathfrak{P}(X) = \mathfrak{P}(X)$$

Therefore, $\mathfrak{P}(X) = L_* \bullet \mathfrak{P}(X)$. Furthermore, according Proposition 3.8 we are found

$$\mathfrak{P}(X) = L_* \bullet \mathfrak{P}(X) = L_* \odot \mathfrak{P}(X)$$

then with consider equivalence 3.2, so that $\mathfrak{P}(X) \in Im\Lambda_{L_*}$.
next, we will be shown $\mathfrak{P}(X)$ is a greatest element in $Im\mathfrak{L}_{K^*} \cap Im\Lambda_{L_*}$ least than X. Given $Y \in Im\mathfrak{L}_{K^*} \cap Im\Lambda_{L_*}$ such that $Y \preceq X$. So that, as claimed by Lemma 2.14 and Proposition 3.8 are found equivalence:

$$Y = K^* \otimes Y = L_* \otimes Y = L_* \bullet Y = L_*(K^* \otimes Y)$$

and because of an asumption $L_* \bullet (K^* \otimes Y) = (L_* \bullet K^*) \otimes Y$. From Definition 2.2, $Y = (L_* \bullet K^*) \otimes Y$ so that $Y \preceq (L_* \bullet K^*) \otimes Y$. According Lemma 2.14, is found $Y = (L_* \bullet K^*)^* \setminus Y$. The mapping $\mathfrak{L}^{\sharp}_{L_* \bullet K^{**}}$ become an isoton mapping, then hold implication $Y \preceq X \Rightarrow (L_* \bullet K^*)^* \setminus Y \preceq (L_* \bullet K^*)^* \setminus X$, which is meant $Y \preceq X$ then $Y = (L_* \bullet K^*)^* \setminus Y \preceq \mathfrak{P}(X)$ □

4.2. **The Greatest Solution in The Inequality of $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ with $\mathbf{K} \in \mathcal{IS}^{n\times n}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$ are a complete idempotent semirings of intervals.** In the Thomas Brunch article [?], S is given semiring. Note that the definition of a closed interval is a set of $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in S | \underline{x} \preceq t \preceq \overline{x}\}$, where $\underline{x} \in S$ and $\overline{x} \in S$ (with $\underline{x} \preceq \overline{x}$) is said lower bound and upper bound of intervals $\mathbf{x}$. The $\underline{x} \in S$ and $\overline{x} \in S$ elements are the upper and lower limits. In order to determine the lower and upper limits, partial order relations need to be completed. A semiring must be an idempotent semiring in order to be provided with a partial order relation. As such, S must be a idempotent semiring.

Definition 4.4. *(**Interval**) Given S an idempotent semiring. A closed interval is set $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in S | \underline{x} \preceq t \preceq \overline{x}\}$, where $\underline{x} \in S$ dan $\overline{x} \in S$ (with $\underline{x} \preceq \overline{x}$) is said lower bound and upper bound of interval* $\mathbf{x}$.

Definition 4.5. *(**Interval Semiring**) An interval set is denoted $\mathcal{IS}$, is equiped with operation algebra element*

$$\mathbf{x}\overline{\oplus}\mathbf{y} \triangleq [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \text{ dan } \mathbf{x}\overline{\otimes}\mathbf{y} \triangleq [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}] \tag{4.4}$$

is semiring, with intervals $\varepsilon = [\varepsilon, \varepsilon]$ and $\mathbf{e} = [e, e]$ are element netral to operation $\overline{\oplus}$ and $\overline{\otimes}$ of $\mathcal{IS}$. Relation order canonical $\preceq_{\mathcal{IS}}$ which is induced of addition such that $\mathbf{x}\overline{\oplus}\mathbf{y} = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \Leftrightarrow \mathbf{x} \preceq_{\mathcal{IS}} \mathbf{y} \Leftrightarrow \underline{x} \preceq_{\mathcal{S}} \underline{y}$ dan $\overline{x} \preceq_{\mathcal{S}} \overline{y}$ where $\preceq_{\mathcal{S}}$ order relation of an idempoten semiring S.

As note, for avoid ambiguity, order relation in $\mathcal{IS}$ will be denoted $\preceq$. Operation 4.4 is an interval which is solid load for all the result of operation, is equal to element which is change of that interval operation.

Given S is a complete idempotent semiring and $\{\mathbf{x}_\alpha\}$ is an uncountable subset of $\mathcal{IS}$, the addition uncountable of element of that subset as follows:

$$\overline{\bigoplus}_\alpha \mathbf{x}_\alpha = [\bigoplus_\alpha \underline{x}_\alpha, \bigoplus_\alpha \overline{x}_\alpha]$$

A top element is set by $\top = [\top, \top]$. If $\mathbf{x}$ and $\mathbf{y}$ are intervals in $\mathcal{IS}$, then $\mathbf{x} \subset \mathbf{y}$ if and only if $\underline{y} \preceq \underline{x} \preceq \overline{x} \preceq \overline{y}$. Furthermore, $\mathbf{x} = \mathbf{y}$ if and only if $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$. An interval such that $\underline{x} = \overline{x}$ is said degenerate. The degenerate interval represented an indeterminate value. In this case, $\mathbf{x}$ is denoted x. The interval semiring $\mathcal{IS}$ is not semifield even if S only one. Of course, exception for the degenerate interval, the interval which is not loaded an invers multiplikatif.

Based on the definition of dual product 3.1, then the definition of dual product $\mathcal{IS}$ is on S is a complete idempotent semiring.

Definition 4.6. ***(Dual Product of Complete Idempotent Semiring $\mathcal{IS}$)*** *In an idempotent semiring of interval, dual product $\odot$ is explained as $\mathbf{x} \odot \mathbf{y} \triangleq [\underline{x} \odot \underline{y}, \overline{x} \odot \overline{y}]$ where $\odot$ is dual product in a complete idempotent semiring S.*

A mapping which preserve order load exsistence over an interval semiring with consider image of an interval bound in the independent direction. Especially aditive closure and $\wedge$ closure could be calculated in efficient direct and are explained as the next :

Proposition 4.7. *Given $\mathcal{IS}$ is a complete idempotent semiring of interval. Aditive closure of matrix $\mathbf{A} \in \mathcal{IS}^{n\times n}$ is given as the following :*

$$\mathbf{K}^* = [\underline{K}, \overline{K}]^* = [\underline{K}^*, \overline{K}^*]$$

and $\wedge$ closure adalah

$$\mathbf{K}_* = [\underline{K}, \overline{K}]_* = [\underline{K}_*, \overline{K}_*]$$

Definition 4.8. ***(Semiring of Pairs)*** [**?**] *Given S a complete semiring. Pair set $(x', x")$ with $x' \in S$ and $x" \in S$ are complete semirings, is denoted $\mathcal{C}(S)$ with $(\varepsilon, \varepsilon)$ elemen nol, (e, e) identity element and $(\top, \top)$ top element. Set of pairs $(x', x")$, $x' \preceq x"$ is complete subsemiring of $\mathcal{C}(S)$, is called $\mathcal{C}_0(S)$*

Proposition 4.9. [**?**] *The Canonical injection $Id_{|\mathcal{C}_0(S)} : \mathcal{C}_0(S) \to \mathcal{C}(S)$ is a residuated mapping and also a dual residuated mapping. Its residual $Id^\sharp_{|\mathcal{C}_0(S)}$ is projector. Its practical calculation as following:*

$$Id^\sharp_{|\mathcal{C}_0(S)}((x', x")) = (x' \wedge x", x") = (\widetilde{x'}, \widetilde{x"}) \tag{4.5}$$

Its dual residual $Id^\flat_{|\mathcal{C}_0(S)}$ also is projector. Its practical calculation as following :

$$Id^\flat_{|\mathcal{C}_0(S)}((x', x")) = (x, x' \oplus x") = (\widetilde{x'}, \widetilde{x"}) \tag{4.6}$$

In the Thomas Brunch article [?], $\mathcal{IS}$ is only an semiring of intervals. whereas $\mathfrak{L}^{\sharp}_{\mathbf{a}}$ is defined so that the lower bound must be found. to find the lower bound, you need a partial order relation. So IS must be an idempotent semiring of intervals.

Proposition 4.10. *[Given $\mathcal{IS}$ is an idempotent semiring of intervals. A mapping $\mathfrak{L}_{\mathbf{k}} : \mathcal{IS} \to \mathcal{IS}$, $\mathbf{x} \mapsto \mathbf{k}\overline{\otimes}\mathbf{x}$ are residuated mappings. Its residual mapping is*

$$\mathfrak{L}^{\sharp}_{\mathbf{k}} : \mathcal{IS} \to \mathcal{IS}, \mathbf{k}\overline{\setminus}\mathbf{x} = [\underline{k} \setminus \underline{x} \wedge \overline{k} \setminus \overline{x}, \overline{k} \setminus \overline{x}]$$

Therefore, $\mathbf{k}\overline{\setminus}\mathbf{l}$ is a greatest solution of $\mathbf{k}\overline{\otimes}\mathbf{x} \preceq \mathbf{l}$, and equal with achieve if $\mathbf{l} \in Im\mathfrak{L}_{\mathbf{k}}$

Proposition 4.11. *Given $\mathcal{IS}$ is an idempotent semiring of interval. For all $\mathbf{k}, \mathbf{l}, \mathbf{o} \in \mathcal{IS}$.*

(1) $(\mathbf{k}\overline{\setminus}\mathbf{l}) \wedge (\mathbf{o}\overline{\setminus}\mathbf{l}) = (\mathbf{k}\overline{\oplus}\mathbf{o})\overline{\setminus}\mathbf{l}$
(2) $(\mathbf{k}\overline{\otimes}\mathbf{l})\overline{\setminus}\mathbf{x} = \mathbf{l}\overline{\setminus}(\mathbf{k}\overline{\setminus}\mathbf{x})$

Analog, can be shown a mapping $\mathfrak{R}_{\mathbf{k}} : \mathcal{IS} \to \mathcal{IS}, \mathbf{x} \mapsto \mathbf{x}\overline{\otimes}\mathbf{k}$ is a residuated mapping.

Proposition 4.12. [?]*A mapping $\wedge_{(k',k")} : \mathcal{C}_0(S), (x', x") \mapsto (k' \odot x', k" \odot x")$ with $(k', k") \in \mathcal{C}_0(S)$ is a dual residuated mapping. Its dual residual as the follows :*

$$\wedge^{\flat}_{(k',k")} : \mathcal{C}_0(S) \to \mathcal{C}_0(S), (x', x") \mapsto (k' \bullet x', k' \bullet x' \oplus k" \bullet x") \tag{4.7}$$

Proof. According Corollary 3.4, a mapping $\wedge_{k',k"} : \mathcal{C}(S) \to \mathcal{C}(S), (x', x") \mapsto (k' \odot x', k" \odot x")$ is a dual residuated mapping and its dual residual $\wedge^{\flat}_{(k',k")} : \mathcal{C}_0(S) \to \mathcal{C}_0(S), (x', x") \mapsto (k' \bullet x', k' \bullet x' \oplus k" \bullet x")$. Pemetaan $\wedge_(k', k")$ preserve order, therefore, $Im\wedge_{(k',k")|\mathcal{C}_0(S)} \in \mathcal{C}_0(S)$. Futhermore, canonical injection $Id_{|\mathcal{C}_0(S)} : \mathcal{C}_0(S) \to \mathcal{C}(S)$ is dual residuated mapping. Therefore, Proposition 2.6 find

$$\left({}_{\mathcal{C}_0(S)|}\wedge_{(k',k")|\mathcal{C}_0(S)}\right)^{\flat} = \left({}_{\mathcal{C}_0(S)|}\wedge_{(k',k")} \circ Id_{|\mathcal{C}_0(S)}\right)^{\flat} = (Id_{|\mathcal{C}_0(S)})^{\flat} \circ (\wedge_{(k',k")})^{\flat} \circ Id_{|\mathcal{C}_0(S)}$$

With the result, Equation 4.6 from Proposition 4.9 find Equation 4.7. □

In the Thomas Brunch article [?], S is a semiring and $\mathcal{IS}$ is only a semiring of interval. whereas $\wedge^{\flat}_{k}$ is defined so that the lower bound must be found. to find the lower bound, you need a partial order relation. So S and $\mathcal{IS}$ must be an idempotent semiring of intervals.

Proposition 4.13. *Given S is a idempotent semirings and $\mathcal{IS}$ is an idempotent semiring of intervals. A mapping $\wedge_{\mathbf{k}} : \mathcal{IS} \to \mathcal{IS}, \mathbf{x} \mapsto \mathbf{k}\overline{\odot}\mathbf{x}$ is a dual residuated mapping. Its dual residual is*

$$\wedge^{\flat}_{\mathbf{k}} : \mathcal{IS} \to \mathcal{IS}, \mathbf{x} \mapsto \mathbf{k}\bar{\bullet}\mathbf{x} = [\underline{k} \bullet \underline{x}, \underline{k} \bullet \underline{x} \oplus \overline{k} \bullet \overline{x}]$$

Therefore, $\mathbf{k} \bullet \mathbf{l}$ is a smallest solution of $\mathbf{k}\overline{\odot}\mathbf{x} \succeq \mathbf{l}$, and equal if $\mathbf{l} \in Im\wedge_{\mathbf{k}}$

Proof. Given $\psi : \mathcal{C}_0(S) \to \mathcal{IS}, (\overline{x}', \overline{x}") \mapsto [\underline{x}, \overline{x}] = [\overline{x}', \overline{x}"]$ is a mapping which mapp order pair in an interval. This mapping is an isomorfism, because that mapping just have relation with bound for handling an interval. Then the result directly according to Proposisi 4.12. □

Corollary 4.14. *Given S idempotent semiring and $\mathbf{K} \in \mathcal{IS}^{n\times p}, \mathbf{X} \in \mathcal{IS}^{p\times q}, \mathbf{Y} \in \mathcal{IS}^{n\times q}$ is matrix. According to Cororally 3.4, a mapping $\wedge_{\mathbf{K}} : \mathcal{IS}^{p\times q} \to \mathcal{IS}^{n\times q}, \mathbf{X} \mapsto \mathbf{K} \odot \mathbf{X}$ is a dual residuated mapping. Its dual residual equal to :*

$$\wedge^{\flat}_{\mathbf{K}} : \mathcal{IS}^{n\times q} \to \mathcal{IS}^{p\times q}, \mathbf{Y} \mapsto \mathbf{K}\bar{\bullet}\mathbf{Y} = [\underline{K} \bullet \underline{Y}, \underline{K} \bullet \underline{Y} \oplus \overline{K} \bullet \overline{Y}] \tag{4.8}$$

According to Proposition 2.11, we can apply in a complete idempotent semiring of intervals as the follows:

Proposition 4.15. *Given $\mathcal{IS}$ is a complete idempotent semiring of intervals, matrices $\mathbf{K}, \mathbf{L}, \mathbf{O} \in \mathcal{IS}^{n\times n}$ and $\mathbf{X} \in \mathcal{IS}^{n\times p}$. The next statements hold :*

$$\mathbf{K}^*\overline{\otimes}\mathbf{K}^\star\overline{\otimes}\mathbf{X} = \mathbf{K}^*\otimes\mathbf{X} \tag{4.9}$$

$$\mathbf{K}^*\overline{\setminus}\mathbf{K}^*\overline{\setminus}\mathbf{X} = \mathbf{K}^*\overline{\setminus}\mathbf{X} \tag{4.10}$$

$$\mathbf{K}^*\overline{\otimes}(\mathbf{K}^*\overline{\setminus}\mathbf{X}) = \mathbf{K}^*\overline{\setminus}\mathbf{X} \tag{4.11}$$

$$\mathbf{K}^*\overline{\setminus}(\mathbf{K}^*\overline{\otimes}\mathbf{X}) = \mathbf{K}^*\overline{\otimes}\mathbf{X} \tag{4.12}$$

$$\mathbf{O}^*\mathbf{K}^* \Leftrightarrow \mathbf{K}^*\mathbf{O}^*\mathbf{X} = \mathbf{K}^*\mathbf{X} = \mathbf{O}^* \setminus (\mathbf{K}^*\mathbf{X}) \Leftrightarrow Im\mathfrak{L}_{\mathbf{K}^*} \subset Im\mathfrak{L}_{\mathbf{C}^*} \Leftrightarrow Im\mathfrak{L}_{\mathbf{K}^*} \subset Im\mathfrak{L}^{\sharp}_{\mathbf{O}^*} \tag{4.13}$$

whereas the dual product, the proposition can be found :

$$\mathbf{L}_*\overline{\odot}\mathbf{L}_*\overline{\odot}\mathbf{X} = \mathbf{L}_*\overline{\odot}\mathbf{X} \tag{4.14}$$

Consequently from Equation 4.14 fullfil the following equivalents :

$$\mathbf{X} = \mathbf{L}_*\overline{\odot}\mathbf{X} \Leftrightarrow \mathbf{X} \in Im\Lambda_{\mathbf{L}_*} \tag{4.15}$$

and the following equivalences hold :

$$\mathbf{X} \succeq \mathbf{K}\overline{\otimes}\mathbf{X} \Leftrightarrow \mathbf{X} = \mathbf{K}^*\overline{\otimes}X \Leftrightarrow \mathbf{X} \preceq \mathbf{K}\overline{\setminus}\mathbf{X} \Leftrightarrow \mathbf{X} = \mathbf{K}^*\overline{\setminus}\mathbf{X} \Leftrightarrow \mathbf{X} \in ImL_{\mathbf{K}^*} \tag{4.16}$$

$$\mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X} \Leftrightarrow \mathbf{L}_*\overline{\odot}\mathbf{X} = \mathbf{X} \Leftrightarrow \mathbf{L}\bar{\bullet}\mathbf{X} \preceq \mathbf{X} \Leftrightarrow \mathbf{L}_*\bar{\bullet}\mathbf{X} = \mathbf{X} \Leftrightarrow \mathbf{X} \in Im\Lambda_{\mathbf{L}_*} \tag{4.17}$$

Based on Proposition 4.10 and Proposition 4.13, the next implications hold :

$$\begin{aligned} \mathbf{X} \in Im\mathfrak{L}_{\mathbf{K}^*} \Rightarrow \mathbf{X} &= [\underline{K}^* \otimes \underline{X}, \overline{K}^* \otimes \overline{X}] = [\underline{K}^* \setminus \underline{X} \wedge \overline{K}^* \setminus \overline{X}, \overline{K}^* \setminus \overline{X}] \\ &= [\underline{K}^* \setminus \underline{X}, \overline{K}^* \setminus \overline{X}] \, since \, \underline{K}^* \otimes \underline{X} \preceq \overline{K}^* \otimes \overline{X} \\ \mathbf{X} \in Im\wedge_{\mathbf{L}_*} \Rightarrow \mathbf{X} &= [\underline{L}_* \odot \underline{X}, \overline{L}_* \odot \overline{X}] = [\underline{L}_* \bullet \underline{X}, \underline{L}_* \bullet \underline{X} \oplus \overline{L}_* \bullet \overline{X}] \\ &= [\underline{L}_* \bullet \underline{X}, \overline{L}_* \bullet \overline{X}] \, since \, \underline{L}_* \odot \underline{X} \preceq \overline{L}_* \odot \overline{X} \end{aligned}$$

According to Proposition 3.9, we can apply in a complete idempotent semiring of intervals $\mathcal{IS}$ as the follows :

Proposition 4.16. *Given a complete idempotent semiring S and matrices $\mathbf{K}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{K}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$. The following statement is equivalent :*

$$\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X} \Leftrightarrow \mathbf{X} \in Im\ \mathfrak{L}_{\mathbf{K}^*} \cap Im\ \Lambda_{\mathbf{L}_*}$$

According to Proposition 4.2, we will apply in a complete idempotent semiring of intervals $\mathcal{IS}$, as the follows :

Proposition 4.17. *Given a complete idempotent semiring S, and matrices $\mathbf{K},\mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{K},\mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X},\mathbf{G} \in \mathcal{IS}^{n\times m}$. The greatest solution $\mathbf{X}$ which hold :*

$$\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X} \text{ and } \mathbf{X} \preceq \mathbf{G}$$

is

$$\widehat{\mathbf{X}} = ((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*)\overline{\setminus}\mathbf{G}$$

with

$$(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*\overline{\setminus}\mathbf{G} = [((\underline{L}_*\bullet\underline{K}^*)^*\setminus\underline{G})\wedge(((\underline{L}_*\bullet\underline{K}^*)\oplus(\overline{L}_*\bullet\overline{K}^*))^*\setminus\overline{G}),((\underline{L}_*\bullet\underline{K}^*)\oplus(\overline{L}_*\bullet\overline{K}^*))^*\setminus\overline{G}]$$

Proof. (1) We are shown $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ dan $\mathbf{X} \preceq \mathbf{G} \Rightarrow \mathbf{X} \preceq \widehat{\mathbf{X}}$. According Proposition 4.16, $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X} \Leftrightarrow \mathbf{X} \in Im\ L_{\mathbf{K}^*} \cap Im\ \Lambda_{\mathbf{L}_*}$. It means $\mathbf{X}$ must hold $\mathbf{X} = \mathbf{L}_* \bullet (\mathbf{K}^* \otimes \mathbf{X})$.

$$\begin{aligned}
\mathbf{X} = \mathbf{L}_*\bar{\bullet}(\mathbf{K}^*\overline{\otimes}\mathbf{X}) \quad &\Leftrightarrow \quad \mathbf{X} = (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\mathbf{X}\ (Proposisi\ 3.6)\\
&\Leftrightarrow \quad \mathbf{X} = ((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*)\overline{\setminus}\mathbf{X}\ (Proposition\ 4.15)\\
&\Leftrightarrow \quad \mathbf{X} = [((\underline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \underline{X}) \wedge ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \overline{X}), ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \overline{X})](Proposition\,4.10)\\
&\Leftrightarrow \quad \mathbf{X} = [((\underline{L}_* \bullet \underline{K}^*)^* \setminus \underline{X}) \wedge (((\underline{L}_* \bullet \underline{K}^*) \oplus (\overline{L}_* \bullet \overline{K}^*))^* \setminus \overline{X}), (((\underline{L}_* \bullet \underline{K}^*) \oplus\\
&\qquad (\overline{L}_* \bullet \overline{K}^*))^* \setminus \overline{X})](Proposition\,4.12\,and\,Proposition\,4.7)
\end{aligned}$$

Futhermore, $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ and $\mathbf{X} \preceq \mathbf{G} \Leftrightarrow \mathbf{X} = [((\underline{L}_* \bullet \underline{K}^*)^* \setminus \underline{X}) \wedge (((\underline{L}_* \bullet \underline{K}^*) \oplus (\overline{B}_* \bullet \overline{A}^*))^* \setminus \overline{X}), (((\underline{L}_* \bullet \underline{K}^*) \oplus (\overline{L}_* \bullet \overline{K}^*))^* \setminus \overline{X})]$ and $\mathbf{X} \preceq \mathbf{G}$. As claimed by Proposition 4.15 $\mathbf{X} = ((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*)\overline{\setminus}\mathbf{X} \Leftrightarrow (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\mathbf{X} \preceq \mathbf{X}$. Because $(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\mathbf{X} \preceq \mathbf{X}$ dan $\mathbf{X} \preceq \mathbf{G}$ then $\mathbf{X} \preceq \widehat{\mathbf{X}} = ((\mathbf{L}_* \bullet \mathbf{K}^*)^*)\overline{\setminus}\mathbf{G}$.

(2) We will be shown $\widehat{\mathbf{X}} \preceq \mathbf{G}$, $\widehat{\mathbf{X}} = \mathbf{K}^*\overline{\otimes}\widehat{\mathbf{X}}$, and $\widehat{\mathbf{X}} = \mathbf{L}_*\overline{\odot}\widehat{\mathbf{X}}$.

First we will be shown $\widehat{\mathbf{X}} \in Im\ \mathfrak{L}_{\mathbf{K}^*}$ which equivalent $\widehat{\mathbf{X}} = \mathbf{K}^*\overline{\otimes}\widehat{\mathbf{X}} = \mathbf{K}^*\overline{\setminus}\widehat{\mathbf{X}}$. According to Proposition 4.15, $\widehat{\mathbf{X}}$ hold

$$(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\widehat{\mathbf{X}} \preceq \widehat{\mathbf{X}} \preceq (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}} \tag{4.18}$$

As a consequence of $\mathfrak{L}_{\mathbf{K}^*}$ an isoton mapping and $\widehat{\mathbf{X}} \preceq (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}}$, then $\mathbf{K}^*\overline{\otimes}\widehat{\mathbf{X}} \preceq \mathbf{K}^*\overline{\otimes}((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}})$. According to Proposition 4.15 is found

$$\mathbf{K}^*\overline{\setminus}\widehat{\mathbf{X}} \succeq \mathbf{K}^*\overline{\setminus}((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}})$$

Furthermore,

$$\begin{aligned}
\mathbf{K}^*\overline{\setminus}((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}}) \quad &= \quad ((\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}}(Proposition\ 4.11)\\
&= \quad (\mathbf{L}_*\bar{\bullet}(\mathbf{K}^*\overline{\otimes}\mathbf{K}^*))\overline{\setminus}\widehat{\mathbf{X}}(Proposition\ 3.6)\\
&= \quad (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}}\\
&= \quad [((\underline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*}) \setminus \underline{\widehat{X}}) \wedge ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*}) \setminus \overline{\widehat{X}}), ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*}) \setminus \overline{\widehat{X}})]\\
&= \quad [((\underline{L}_* \bullet \underline{K}^*) \setminus \underline{\widehat{X}}) \wedge (((\underline{L}_* \bullet \underline{K}^*) \oplus (\overline{L}_* \bullet \overline{K}^*)) \setminus \overline{\widehat{X}}), (((\underline{L}_* \bullet \underline{K}^*) \oplus\\
&\qquad (\overline{L}_* \bullet \overline{K}^*)) \setminus \overline{\widehat{X}})]
\end{aligned}$$

So we are found $\mathbf{K}^*\overline{\setminus}\widehat{\mathbf{X}} \succeq (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}}$. As claimed by an Inequality 4.18, $(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}} \succeq \widehat{\mathbf{X}}$. As a result, $\mathbf{K}^*\overline{\setminus}\widehat{\mathbf{X}} \succeq (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\setminus}\widehat{\mathbf{X}} \succeq \widehat{\mathbf{X}}$. So we are found

$\mathbf{K}^*\backslash\widehat{X} \succeq \widehat{X}$. So that, $\widehat{X} \succeq K^*\backslash\widehat{X}$ (because $K^* \succeq E$) then $K^*\backslash\widehat{X} = \widehat{X}$, i.e. $\widehat{\mathbf{X}} \in Im\,\mathbf{K}^*$.

The second step, we are shown $\widehat{\mathbf{X}} \in Im\,\Lambda_{\mathbf{L}_*}$, i.e. $\widehat{\mathbf{X}} = \mathbf{L}_*\overline{\odot}\widehat{\mathbf{X}} = \mathbf{L}_*\bar{\bullet}\widehat{\mathbf{X}}$. From an Inequality 4.18 is found $\widehat{\mathbf{X}} \succeq (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\widehat{\mathbf{X}} = \mathbf{L}_*\bar{\bullet}(\mathbf{K}^*\overline{\otimes}\widehat{\mathbf{X}}) = \mathbf{L}_*\bar{\bullet}\widehat{\mathbf{X}}$ (because $\widehat{\mathbf{X}} = \mathbf{K}^*\overline{\otimes}\widehat{\mathbf{X}}$). in the other side, $\mathbf{L}_* \preceq \mathbf{E}^{\odot}$. As a consequence of $\Lambda_{\mathbf{L}_*}$ an isoton mapping, then $\mathbf{L}_*\overline{\odot}\widehat{\mathbf{X}} \preceq \mathbf{E}^{\odot}\overline{\odot}\widehat{\mathbf{X}}$. Therefore, As claimed by Theorem 2.4 we are found $\widehat{\mathbf{X}} \preceq \mathbf{L}_*\bar{\bullet}\widehat{\mathbf{X}}$. So that $\widehat{\mathbf{X}} = \mathbf{L}_*\bar{\bullet}\widehat{\mathbf{X}} = \mathbf{L}_*\overline{\odot}\widehat{\mathbf{X}}$.

The last step because $(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^* \succeq \mathbf{E}$, then $(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*\overline{\odot}\mathbf{G} \succeq \mathbf{E}\overline{\odot}\mathbf{G}$. According to Theorem 2.3, we are found $(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*\overline{\backslash}\mathbf{G} \preceq \mathbf{G}$. So $\widehat{\mathbf{X}} \preceq \mathbf{G}$

□

Theorema which is explained Hardouin [?], he given that S is just semiring. But, when we proof, we need S complete idempotent semiring. Because that is Klenee Star which is defined Proposisi 4.7 that must defined over infinite sum.

Proposition 4.18. *Given S complete idempotent semiring, and $\mathbf{K}, \mathbf{L} \in \mathcal{IS}^{n\times n}, \mathbf{X} \in \mathcal{IS}^{n\times m}$. If $\forall \mathbf{X}$ equation $\mathbf{L}_*\bar{\bullet}(\mathbf{K}^*\overline{\otimes}\mathbf{X}) = (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\otimes}\mathbf{X}$ hold, a mapping*

$$\mathfrak{P} : \mathcal{IS}^{n\times m} \to \mathcal{IS}^{n\times m}, \mathbf{X} \mapsto (\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)\overline{\backslash}\mathbf{X}$$

with

$$(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*\overline{\backslash}\mathbf{X} = [((\underline{L}_*\bullet\underline{K}^*)^*\backslash\underline{X})\wedge(((\underline{L}_*\bullet\underline{K}^*)\oplus(\overline{L}_*\bullet\overline{K}^*))^*\backslash\overline{X}), ((\underline{L}_*\bullet\underline{K}^*)\oplus(\overline{L}_*\bullet\overline{K}^*))^*\backslash\overline{X}]$$

is a projector in $Im\,\mathfrak{L}_{\mathbf{K}^*} \cap Im\,\wedge_{\mathbf{L}_*}$, officially,

$$\mathfrak{P}(X) = \{\bigvee \mathbf{Y} | \mathbf{Y} \preceq_{\mathcal{IS}} \mathbf{X}\,and\,\mathbf{Y} \in Im\,\mathfrak{L}_{\mathbf{K}^*} \cap Im\,\wedge_{\mathbf{L}_*}\}$$

Proof. Proposition 4.18 is aplication of Proposition 4.3. In practice its calculation, from Proposition 4.10, we find

$$(\mathbf{L}_*\bar{\bullet}\mathbf{K}^*)^*\overline{\backslash}\mathbf{X} = [((\underline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \underline{X}) \wedge ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \overline{X}, ((\overline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* \setminus \overline{X})]$$

with according Proposition 4.12 and Proposition 4.7 are found

$$(\underline{\mathbf{L}_*\bar{\bullet}\mathbf{K}^*})^* = (\underline{L}_* \bullet \underline{K}^*)^*$$

and

$$\overline{(\mathbf{K}_*\bar{\bullet}\mathbf{K}^*)}^* = ((\underline{L}_* \bullet \underline{K}^*) \oplus (\overline{L}_* \bullet \overline{K}^*))^*$$

□

5. Conclusion

(1) There is a the greatest solution have owned an inequality solution of $K \otimes X \preceq X \preceq L \odot X$ and $X \preceq G$ with matrix K, L which is element of the complete idempotent semiring $S^{n\times n}$ dan X, G which is element of a complete idempotent semiring $S^{n\times m}$.

(2) There is a sufficient condition for the projector have owned an inequality solution of $K \otimes X \preceq X \preceq L \odot X$ with matrix K, L which is element of the complete idempotent semiring $S^{n\times n}$ dan X, G which is element of a complete idempotent semiring. Futhermore, the projector is the greatest solution of inequality $K \otimes X \preceq X \preceq L \odot X$.

(3) Based on previous research, the greatest solution to the inequality of $\mathbf{K}\overline{\otimes}\mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L}\overline{\odot}\mathbf{X}$ and $\mathbf{X} \preceq \mathbf{G}$ can be applied to the matrix $\mathbf{K}, \mathbf{L}$ which which is element of the complete idempotent semiring of intervals $\mathcal{IS}^{n\times n}$ and $\mathbf{X}, \mathbf{G}$ which which is element of the complete idempotent semiring of intervals $\mathcal{IS}^{n\times m}$.

(4) There are sufficient conditions for the projector to have an inequality solution $\mathbf{K} \otimes \mathbf{X} \preceq \mathbf{X} \preceq \mathbf{L} \odot \mathbf{X}$ and $\mathbf{X} \preceq \mathbf{G}$ with the matrix $\mathbf{K}, \mathbf{L}$ which is element of the complete idempotent semiring of intervals $\mathcal{IS}^{n\times n}$ and $\mathbf{X}, \mathbf{G}$ which which is element of the complete idempotent semiring of intervals $\mathcal{IS}^{n\times m}$ from the results of previous studies.

ACKNOWLEDGMENT

Researcher fully funded by Universitas PGRI Adi Buana Surabaya.

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