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ABSTRACT REAL NUMBERS

(BILANGAN REAL ABSTRAK)

SOEPARNA DARMAWIJAYA

Abstract. In this paper we represent a result of our study in field of analysis , that is, an abstraction of the system of real numbers. We start by defining a set of positive elements in a countable infinite (denumerable) field and, hence, we obtain a linearly ordered field which we call a field of rational elements or a rational field. After that we may introduce irrationals elements in our rational field. And, at last we have a system of real abstract numbers.

Keywords:

1 A RATIONAL FIELD

We remind that an algebraic structure $\mathcal{F}$ is called a field if

(i). $\mathcal{F}$ is a commutative group with respect to binary addition operation $(+)$; the null element is denoted by 0,

(ii). $\mathcal{F} - \{0\}$ is a group with respect to binary product operation $(\cdot)$; the unit element is denoted by 1,

(iii). $x \cdot 0 = 0 \cdot x = 0$ for every $x \in \mathcal{F}$ and $\mathcal{F}$ has distributive property:

$$x \cdot (y + z) = x \cdot z + y \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z$$

for every $x, y, z \in \mathcal{F}$.

In what follows we only consider that our field $\mathcal{F}$ has a countable infinite (denumerable) elements and $\mathcal{F} - 0$ is a commutative group with respect to binary product operation $(\cdot)$, that is

$$x \cdot y = y \cdot x$$

for every $x, y \in \mathcal{F}$.

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Definition 1.1. ([5]) *Let $\mathcal{F}$ be a field. A set P of $\mathcal{F}$ is called the* ***set of positive elements*** *if:*

(i). $0 \notin P$,

(ii). $x, y \in P \Rightarrow x + y \in P$,

(iii). $x, y \in P \Rightarrow x \cdot y \in P$,

(iv). If $x \in \mathcal{F}$ then only one of the following statements is true:

$$x \in P, \ -x \in P, \ or \ x = 0.$$

If $x \in P$ then x is called a ***positive element*** *and if $-x \in P$ then x is called a* ***negative element****.*

Theorem 1.2. *If $x \in \mathcal{F}$ and $x \neq 0$ then $x^2 \in P$, especially $1 \in P$*

Proof. $x^2 = x \cdot x = -x \cdot -x \in P$, especially $x = 1$. □

Definition 1.3. *Let $\mathcal{F}$ be a field. If $x, y \in \mathcal{F}$ we define*

$$x < y \Leftrightarrow y - x \in P.$$

Theorem 1.4. *Let $\mathfrak{F}$ be a field. For every $x, y \in \mathcal{F}$ then only one of the following statements is true:*

$$x < y, \ y < x, \quad or \quad x = y.$$

Proof. By Definition 1.3, it is true that

$$x < y, \ y < x, \quad \text{or} \quad x = y \ \Leftrightarrow \ y - x \in P, \ x - y \in P \ \text{ or } \ x - y = 0.$$ □

Corollary 1.5. *Every filed $\mathcal{F}$ is a linearly ordered set with respect to binary relation "$<$" then $\mathcal{F}$ is called a linearly ordered field or a* ***rational field*** *and its members are called* ***rational elements****.*

In order in speaking of our rational field $\mathcal{F}$ more detail and more clear we shall rewrite in writing its elements as follows. $x, y, \cdots$ denote the positive element and $-x, -y, \cdots$ denote the negative elements. Since $\mathcal{F} - \{0\}$ is commutative group with respect binary opreation $(\cdot)$ then every its element has an inverse element. $x^{-1} = \frac{1}{x}$ is the inverse element of x and $(-x)^1 = \frac{1}{-x} = -\frac{1}{x}$ is the inverse element of $-x$. Therefore we have the following formulas:

$$\frac{x}{y} = x \cdot \frac{1}{y} = x \cdot y^{-1} \text{ and } \frac{x}{-y} = -x \cdot \frac{1}{y} = -x \cdot y^{-1} = \frac{-x}{y},$$

$$1 = (x \cdot y)(x \cdot y)^{-1} = (x \cdot y)\frac{1}{x \cdot y} = (x \cdot y)\frac{1}{x} \cdot \frac{1}{y} \ \Rightarrow \ (x \cdot y)^{-1} = \frac{1}{x \cdot y} = \frac{1}{x} \cdot \frac{1}{y},$$

$$\frac{x}{y} = \frac{x \cdot v}{y \cdot v} = \frac{x \cdot -v}{y \cdot -v} \text{ for every } v, \frac{x}{y} \cdot \frac{u}{v} = \frac{x \cdot u}{y \cdot v},$$

$$\frac{x}{y} + \frac{u}{v} = \frac{x \cdot v}{y \cdot v} + \frac{y \cdot u}{y \cdot v} = \frac{x \cdot v + y \cdot u}{y \cdot v} \text{ and } \frac{x}{y} - \frac{u}{v} = \frac{x \cdot v}{y \cdot v} - \frac{y \cdot u}{y \cdot v} = \frac{x \cdot v - y \cdot u}{y \cdot v},$$

$$\frac{x}{y} = \frac{u}{v} \;\Leftrightarrow\; x \cdot v = y \cdot u,$$

$$0 = \frac{0}{y} \notin P,\; \frac{x}{y}, \frac{u}{v} \in P \;\Rightarrow\; \frac{x}{y} + \frac{u}{v},\; \frac{x}{y} \cdot \frac{u}{v} \in P,$$

$$\frac{x}{y} < \frac{u}{v} \;\Leftrightarrow\; \frac{x \cdot v}{y \cdot v} < \frac{y \cdot u}{y \cdot v} \;\Leftrightarrow\; \frac{x \cdot v < y \cdot u}{y \cdot v} \;\Leftrightarrow\; \frac{u}{v} - \frac{x}{y} \in P.$$

Since our rational field $\mathcal{F}$ is a linearly ordered set we shall show that the binary relation "=" is an equivalent relation.

Theorem 1.6. *The binary relation "=" is an equivalent relation in our rational field* $\mathcal{F}$*.*

(i). "=" is reflexive since

$$\frac{x}{y} = \frac{x}{y} \text{ for every } \frac{x}{y} \in \mathcal{F}.$$

(ii). "=" is transitive:

$$\frac{x}{y} = \frac{u}{v} \text{ and } \frac{u}{v} = \frac{w}{z} \;\Rightarrow\; \frac{x}{y} = \frac{w}{z}$$

since

$$x \cdot v = y \cdot u \text{ and } u \cdot z = v \cdot w \;\Rightarrow\; x \cdot v \cdot z = y \cdot v \cdot w \;\Rightarrow\; x \cdot z = y \cdot w$$

(iii). "=" is symmetric since

$$\frac{x}{y} = \frac{u}{v} \;\Leftrightarrow\; x \cdot v = y \cdot u \;\Leftrightarrow\; \frac{u}{v} = \frac{x}{y}.$$

Corollary 1.7. *Every rational field splits into disjoint posets; every poset contains every its two element are equivalent and the set of all posets is a linearly ordered set.*

2 A SYSTEM OF REAL ELEMENTS (A SYSTEM OF ABSTRACT NUMBERS)

Definition 2.1. ([3]) *Let* $\mathcal{F}$ *be rational field. If* $A, B \subset \mathcal{F}$ *such that*

$$A \cup B = \mathcal{F} \text{ and } A \cap B = \phi$$

and every element of B *is less than every element of* A *then the pair* B/A *is called a* ***cut*** *of* $\mathcal{F}$*.*

Further, if B/A *and* B_1/A_1 *are two cut of* $\mathcal{F}$ *we say* B/A *is less than* B_1/A_1 *and we write*

$$B/A < B_1/A_1$$

if $B \subset B_1$ *and* $B_1 - B \neq \varphi$*.*

Theorem 2.2. *If* B/A *a cut of a rational field* $\mathcal{F}$ *then only one of the following statements is true:*

(i). A has a least element and B has no a largest element.
(ii). A has no a least element and B has a largest element.
(iii). That is, there is a gap between B and A.

Proof. Since our rational field $\mathcal{F}$ only has a coutable infinte elements , $\mathcal{F}$ is linearly ordered set, and element of B is less than every element of A then only one of the following statents is true :
(i) A has a least element and B has no a largest element. It means that if α is the least element of A then $beta < \alpha$ for every $\beta \in B$
(ii) A has no a least element and B has a largest element. It means that if γ is the largest element of B then $\gamma < \delta$ for every $\delta \in A$. □

Since if B/A is a cut of a rational field $\mathcal{F}$ there is a gap between B and A we want to fill the gap by an entity $\rho(B/A)$ which we call it an **original irrational element**.

Definition 2.3. *Let B/A be a cut of a rational field $\mathcal{F}$.*

(i). If B has some positive elements we define an entity $\rho(B/A)$ which satisfies the following property:

$$-\frac{x}{y}, \frac{u}{v} < \rho(B/A) < \frac{w}{z}$$

for every $-\frac{x}{y}, \frac{u}{v} \in B$ and $\frac{w}{z} \in A$.

(ii). If A has some negative elements we define an entity $\rho(B/A)$ which satisfies the following property:

$$-\frac{x}{y} < -\rho(B/A) < -\frac{u}{v}, \frac{w}{z}$$

for every $-\frac{x}{y} \in B$ and $-\frac{u}{v}, \frac{w}{z} \in A$.

$\rho(B/A)$ is called a (positive) ***original irrational element.***

Straightly, by the Definition 2.3 be have the following theorem.

Theorem 2.4. *If $\mathcal{R}$ is the collection of all elements of a rational field $\mathcal{F}$ and all original irrational elements then $\mathcal{R}$ is a completely linearly ordered set. Hence $\mathcal{R}$ is a completely ordered field with respect to binary operations "+" and "·" and linear ordering "¡" . Every increasing (decreasing) sequence bounded above (bellow) in $\mathcal{R}$ has supremum (infimum).*

Let B/A be a cut of a rational field with B have some positive elements, and hence, every element of A is positive. We define:

$$-A = \{-\frac{x}{y}; \frac{x}{y} \in A\} \text{ and } -B = (-A)^c.$$

Then we have $-A/-B$ is a cut of $\mathcal{F}$ and $-B$ has some negative elements. Therefore we have the following theorem.

Theorem 2.5. *Let B/A be a cut of a rational field $\mathcal{F}$ with B has some positive elements. Then:*

$$-\frac{x}{y}, \frac{u}{v} < \rho(B/A) < \frac{w}{z}$$

for every $-\frac{x}{y}, \frac{u}{v} \in B$ and $\frac{w}{z} \in A$ if and only if

$$-\frac{w}{z} - A < -\rho(A/B) < \frac{x}{y}, -\frac{u}{v}$$

for every $-\frac{w}{z} \in -A$ and $\frac{x}{y}, -\frac{u}{v} \in -B$.

By Theorem 2.4 and Theorem 2.5 we may the following conclusion since:

(i). $\mathcal{R}$ is a field and it can be considered as a generalization of a rational field $\mathcal{F}$; and we call $\mathcal{R}$ is a **system of abstract real numbers.** Every member of $\mathcal{R}$ is called an **abstract real number**. Every member of $\mathcal{R}$ is called an **abstract real number**.

(ii). $\mathcal{R}$ is symmetric with respect to 0 (zero number):

$$\frac{x}{y},\ \rho(B/A) \text{ are positive abstract real numbers}$$
$$\Leftrightarrow\ -\frac{x}{y},\ -\rho(B/A) \text{ are negative abstract real numbers.}$$

(iii). Every member of $\mathcal{R}$ is called an **abstract real number**, especially

$$\pm\frac{x}{y} \pm \frac{u}{v}\rho(B/A).$$

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SOEPARNA DARMAWIJAYA
Department of Mathematics, Universitas Gadjah Mada
soeparna@ugm.ac.id