COGNITIVE ARITHMETIC:
Mental Processing of Addition and Multiplication

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In contrast to the many studies of language processing, there have been relatively few studies of arithmetic processing in cognitive psychology. Authors of textbooks for university students, such as Solso (1991), do not appear to feel a need to address cognitive arithmetic issues in their books. Some of the main reasons for the lack of interest, courage, and research effort might include beliefs that language is more important than arithmetic and the greater number of varieties in language than in arithmetic.

Cognitive arithmetic is a rich domain in which arbitrary notations and symbols are manipulated and processed internally according to both innate and learned rules. Until now most studies have dealt with simple cognitive arithmetic processing, especially addition and multiplication operations. Subtraction and division operations can hardly be found except in a subtraction study nested in a series of arithmetic experiments (i.e. Krueger & Hallford, (1988). The strategies that children and adults employ in simple addition and multiplication were first investigated by Groen and Parkman (1972) and most recently by LeFevre, Bisanz, and MrKonjic (1988). It seems that research in the more complex arithmetic operations has not been an interesting area for most cognitive psychologists.

Most of the researchers have used response time measures and error rates to test various models of cognitive arithmetic processing in children and adults. In addition, a series of polemics of how children and adults process basic number between Baroody (1985) and Ashcraft (1985) give a clear evidence that cognitive arithmetic is an extensive as well as a challenging area needs investigating. Arithmetic, as language, is a symbolic activity that is well learned and is used extensively in everyday life (Zbrodoff & Logan, 1986). Additionally, arithmetic is attractive theoretically because of several alternative models underlying its process. This paper discusses two major models that have attracted many researchers for more than two decades. The first model is counting-based models, based primarily on the work of Groen and Parkman (1972), Parkman (1972), Parkman and Groen (1971), and Restle (1970). The second model is network-retrieval model which based largely on the studies by Ashcraft and Battaglia (1978), Ashcraft and Stazyk (1981), and Stazyk, Ashcraft, and Hamann (1982).

Counting-Based Model

Groen and Parkman (1972) and Parkman and Groen (1971) proposed a model of how children and adults solved simple addition problems (single digit 0-9 for addends). The model, the counting-based model, assumes the existence of a counter with two important operations. The first operation is setting the counter to the specified number, i.e. one addend. The second operation is incrementing the value of the counter by one successively for a number of times, i.e. the other addend.

After testing and comparing five variations of the counting-based model, Groen and Parkman (1972) concluded that the best predictor for the model was the minimum addend. Total response time for the counter to perform the operations was time to set the counter and the multiplication of time to increment the counter and minimum addend.

Parkman and Groen (1971), using college students, investigated their performance on simple addition problems (each problem consisted of 2 addends, each was one-digit and nonnegative number, and their sum, two-digit number). The results showed that: (1) there was a significant difference between false negative response time and false positive response time; the false positive response was faster than the false negative one, (2) response times varied as a function of the minimum digit of the two addends, and (3) response times varied as a function of the sum of the two addends.

The most striking result of the above study is a strong relationship between the size of the minimum addend and the response time required for verification of the sum. This finding supports their theoretical explanation that people employ unconscious counting process; first, the maximum addend is determined, and then the value is incremented by ones for a number of times equal to the minimum addend.

In addition to the important finding, Parkman and Groen (1971) concluded that the procedure used to solve the addition problems had four successive stages. First, a preprocessing stage. Second, an addition stage, where the sum of the two addends was computed. Third, a comparison stage, where the result of the addition stage was compared to the stated or presented sum. Fourth, a response execution stage. The first and fourth stages are considered important by the researchers on the basis that the effects of the stages are constant and non interactive with the other two stages. With slightly different in the second and the third stages, Parkman (1972) stated that the stages could be applied in multiplication operations.

In the light of defending the strength of the minimum addend factor in counting-based model, Winkelman and Schmidt (1974) investigated the effects of associative interference of single-digit addition. They showed a significant difference between associative confusions (e.g. 3 + 3 = 9) and nonassociative confusions (e.g. 3 + 3 = 12). The response time of addition operation for associative confusion was 54 msec. slower than the comparable response time for nonassociative confusion (749.5 msec. versus 695.5 msec.). It is apparent that the finding rules out the explanation that simple arithmetic operation based solely on internal computations and suggests the involvement of associative components in the operation.

In order to avoid the technical difficulties of speed of adding measurement in the counting-based model, such as verbal response timing and subject's tendency to stretch out complex numerical responses, Restle (1970) presented a pair of two numbers and their sum in an addition-verification task. Subject only had to decide whether the sum was true or false. Results of the experiment indicated that when the base numbers were close to 100 (e.g. 99 and 101), the response time for correct responses was faster than when the base numbers were farther from 100 (e.g. 83 and 117). It seems that high speed of addition process for base numbers closed to 100 depends on the 'approximation and correction' phenomenon. Borrowing an example from the experiment, when given a problem of 63 + 34, a subject may compute 60 + 30 = 90 as an approximation and then makes a correction by adding the second digits of the two addends, 3 + 4.

In the light of arithmetic operation processes, it is reasonable to infer that subjects in Restle study had to perform three steps of processing. First, rounding two addends to the nearest ten (e.g. 23 becomes 20 and 27 becomes 30). Second, approximating the sum of the two addends based on the rounding numbers. Third, making correction by adjusting the difference between the original numbers with their roundings.

Network-Retrieval Model

Even though two predictors of the counting-based model, the minimum addend and the correct sum, might be superior for the response time in addition performance, Ashcraft and Battaglia (1978) claimed the counting-based model was flawed because of the 'ties' problem. In the meantime, ties problem, where addend 1 is equal to addend 2, does not have any significant linear effect in most equations (see Groen & Parkman's experiment, 1972). However, the main role of the ties appears to be a constant value. It could be noted that the ties should give a substantial effect, since its characteristics are very different from the nonties ones (addend 1 is smaller or bigger than addend 2).

Based on the common standpoint, it can be predicted that the response time for the ties addition is slower than the response time for the nonties addition. That is why Ashcraft and Battaglia (1978), Cambell (1987a, 1987b), and Stazyk, Ashcraft, and Hamann (1982) proposed that the performance of addition-verification task might depend on access to memorized information in a network-retrieval basis.

Based on two experiments aimed to test the network-retrieval model, the proponents of the model concluded that simple mental addition was largely a memory retrieval phenomenon. They claimed that the network-retrieval model was the best in giving explanation of mental addition performance in the adults. They give two reasons based on a series of intensive experiments. First, the network- retrieval model generates some specific predictions about response time performance, such as priming or repetition effects in both verification and production tasks. For example, given a second stimulus of 5 + 8 = 13 after a first stimulus of 6 + 7 = 13, the model predicts that it will be a speeded verification response time because of the repetition of the sum. In another example, given a second stimulus of 7 + 5 = 12 after a first stimulus of 7 + 6 = 13, the model predicts that it will be another speeded verification response time because of the repetition of the second addend.

Second, one of the important findings indicates that false negative and positive response times are slower when the split is small than when the split is large (see Ashcraft & Battaglia, 1978). It means that the sum stated in addition problem first compared with the sum retrieved from the long-term memory. In the case that the difference between the stated sum and the retrieved sum is small, it seems there is a repetition process, that takes longer time, in the memory to check the difference whether it is true or false.

One of the other weaknesses of the counting-based model pointed out by the proponents of the network-retrieval model is the generality of the interpretation when applied to incorrect operation (see Ashcraft & Battaglia, 1978). It means that the split of two addends is an important factor to differentiate the performance of the small split and the larger split. Ashcraft and Battaglia (1978) solved the split problem in their experiments by presenting two kinds of stimuli. The first ones were the reasonable-false stimuli (stated sums were incorrect by 1 or 2 splits) and the second ones were the unreasonable-false stimuli (stated sums were incorrect by 5 or 6 splits).

The data indicated that response time of the reasonable problem was best predicted by the square of the problem's correct sum. They explained the result as indication of two important points. First, the addition operation problems with large sums became increasingly difficult to verify. Second, the eventual verification depended on retrieval of the correct answer from memory. It can be argued that the split problem has not been solved properly in the Ashcraft and Battaglia's study. It is apparent that they give no clear cut solution whether response time to perform unreasonable-false addition operation is significantly different from the reasonable-false addition operation. However, the data indicated that with ties problem excluded, false response time was much higher if the split between the stated sum and the correct sum was a reasonable answer of 1 or 2 rather than an unreasonable answer of 5 or 6 (1,144 msec. versus 1,044 msec.). In addition, response time for the correct sum was the lowest of all, 1,016 msec.

Discussion

Numerical System

Both models, the counting-based and the network-retrieval, do not account for other notational systems. All of their studies have used Arabic numeral system as stimuli. One can ask a question whether their findings can be applied to other numeral systems such as Roman.

Gonzales and Kolers (1982) tested the idea that the mental operation on symbols from different notational systems would depend on the interpretation of the symbols and notational characteristics. In the experiment, the main task was mental addition and the stimuli were equations of Roman numerals (e.g. I, II, III, IV) and Arabic numerals (e.g. 1, 2, 3, 4). The results showed that response time depended on the notational characteristics of the symbols, Roman and Arabic, and also depended on the quantities. In general, the fastest response time for false positive and false negative instances were condition AAA (Arabic + Arabic = Arabic) and the slowest ones were condition RRR (Roman + Roman = Roman) with other conditions were in between.

It could be noted that the position of the Roman numerals is important in the slowing performance. A Roman numeral as an addend slows performance rather than a Roman numeral as a sum. In other word, condition RRA (Roman + Roman = Arabic) takes longer response time than condition AAR (Arabic + Arabic = Roman) or RAR (Roman + Arabic = Roman) and condition AAR takes less response time than conditions ARA or RAA. The outcomes suggest that position is important because the Roman symbols are more disruptive when they stand for addends than when they stand for total or sum of addends (Gonzales & Kolers, 1982).

The second important result indicated that response time increased with an increase in the number of Roman numerals. For example, a Roman numeral added to a larger Arabic numeral was responded more quickly than the reverse. However, response times were faster when the minimum addend was Roman numeral than when it was an Arabic numeral. It seems that subjects make strategy of mental operation by using analog counting, like tallying, for the Roman symbols.

The different response times for the mental operation on Roman and Arabic numerals can be attributed to the characteristics of the notational system. The Roman system has the analog counting system that represents one-to-one relationship between the base number (I) and the larger number (e.g. II and III) and has the grouping for five (V) and ten (X). On the other hand, the Arabic system lacks the property of analogy between the quantity and its symbol represented (Gonzales & Kolers, 1982).

Gonzales and Kolers study leads us to raise the question of whether changes of the symbols influence the mental operation. This problem is much more complicated since most of the studies in cognitive arithmetic have used Arabic numerals as stimuli. Mental operations to this kind of symbols are highly overtrained since the childhood in most parts of the world. Roman numerals, in contrast, have been used largely for labels, such as numbers of front pages in textbooks.

Split and Response Time

Another issue that has not been addressed in both models of cognitive arithmetic is the relationship between split and response time. Even though Parkman and Groen (1971) found that the false responses were slightly faster in the case of split 1 than in the case of 2 (777 msec. versus 788 msec.), they did not explain clearly the process underlying the relationship.

Krueger and Hallford (1984) pointed out that even though split effect or 'symbolic distance effect' was an important factor in the prediction of response time performance, it was failed account for a decrease in false response time when the split was increased from 1 (e.g. 2 + 2 = 5) to (e.g. 2 + 2 = 6).

Odd-Even Rule

Another important criticism, especially to the network-retrieval model, is the odd-even rule issue. Krueger and Hallford (1984) argued that some of the concepts in Ashcraft and Battaglia study violated the odd-even rule in sum verification task. The odd-even rule states that if only one of the two addends is odd, then the correct sum must be odd (i.e. odd + even = odd or even + odd = odd), and other combinations of two addends must be even (i.e. odd + odd = even and even + even = even). The odd-even rule is violated whenever the stated sum deviates from the correct sum, whether odd or even, by split of odd (Krueger, 1986; Krueger & Hallford, 1984).

Krueger and Hallford (1984), using 24 university students in a true-false verification task, investigated whether they used the odd-even rule in false sum stimuli. The experimenters created splits of 1, 2, 3, and 4 with the correct sums ranging from 10 to 17. The results of experiment 1 indicated that (1) response time of correct sum was faster than that of false sum but slightly less accurate (1,205 msec. versus 1,367 msec.), (2) response time of correct sum increased significantly as the size of the minimum addend increased from 0 to its minimum possible value, (3) the reduction in response times and errors was accompanied by downward dips at splits of +1 and +3, and (4) there was no difference between a split of +1 and a split of +2 and also there was no difference between a split of +3 and a split of +4.

In multiplication operation, the odd-even rule should be easier to apply than in addition operation because people do not need to compare the two multipliers in order to use the rule (see Krueger, 1986). We can watch mainly for evenness since it is more important than oddness. Detecting an even multiplier indicates that the product must be even, whereas the two multipliers must be odd to get the odd product.

Autonomy of Arithmetic Processing

One of the most striking result in Krueger and Hallford (1984) study is the claim from most subjects that they were unaware of having used the odd-even rule. Apparently subjects used the rule in an autonomous processing fashion. This suggests another issue whether people perform arithmetic operations autonomously. The issue have been tested in a series of experiments by Zbrodoff and Logan (1988). In the study, the researchers found that simple arithmetic processes were partially autonomous. In four of six experiments, the results ruled out the possibility that arithmetic processes were not autonomous by showing that subjects produced a Stroop-like associative confusion effect.

However, the remaining experiments did not support the possibility that arithmetic processes were completely autonomous by showing that subjects could inhibit the processes at will.

The conclusions, even though rather ambiguous, suggest an important issue in cognitive arithmetic. If the processes underlying simple arithmetic operations partially autonomous, it might be hypothesized that some simple arithmetic operations, such as addition and multiplication of two single-digit numbers, could begin without intention.

In a slightly different term, LeFevre, Bisanz, and MrKonjic (1988) addressed a question of whether activation of arithmetic operations was automatic or obligatory. They presented subjects a pair of numbers and a third number as a probe. The result indicated that sum probes were rejected more slowly than neutral probes (855 msec. versus 889.). This finding clearly supports a prediction that arithmetic operation is obligatory because activation of arithmetic facts occurs upon presentation of the pair of numbers.

Easy and Difficult Tasks

Another issue in cognitive arithmetic is type of tasks, i.e. easy or difficult, in most of the addition and multiplication experiments. Campbell and Graham (1985), comparing multiplication performance between children in Grade 2 to 5 and undergraduates and graduates students, found two main results. First, children made errors of 12.5 per cent on easy task (multiplication of small numbers) and errors of 49.4 per cent on difficult task (Multiplication of large numbers). Second, students made errors of 5.63 per cent on easy task and errors of 13.70 per cent on difficult task.

The difference between error rates on the easy task and error rates on the difficult task for both children and adults seems reflect the basic problem of size effect in arithmetic operations. It means that large number combinations tend to be more difficult than small number combinations. The conclusion is supported by a related study comparing a verification task and a production task in multiplication (Campbell (1987b). The researcher found that response times for both tasks were much faster for the easy problems than for the difficult ones. Response time in production task for the easy problems was 798 msec. whereas for the difficult problems was 998 msec. In the verification task, response time for the easy problems was 928 msec. whereas for the difficult problems was 1,094 msec.

References

Ashcraft, M.H., (1985). Is it farfetched that some of us remember our arithmetic facts? *Journal of Research in Mathematics Education,* 16, 99-105.

Ashcraft, M.H., & Battaglia, J. (1978). Cognitive arithmetic: evidence for retrieval and decision processes in mental addition. *Journal of Experimental Psychology: Human Learning and Memory,* 4, 527-538.

Ashcraft, M.H., & Stazyk, E.H. (1981). Mental addition: a test of three verification models. *Memory & Cognition,* 9, 185-196.

Baroody, A.J. (1985). Mastery of basic number combinations: internalization of relationships or facts? *Journal of Research in Mathematics Education,* 16, 83-98.

Campbell, J.I.D. (1987a). Network interference and mental multiplication. *Journal of Experimental Psychology: Learning, Memory, and Cognition,* 13, 109-123.

Campbell, J.I.D. (1987b). Production, verification, and priming of multiplication facts. *Memory & Cognition*, 15, 349-364.

Campbell, J.I.D., & Graham, D.J. (1985). Mental multiplication skill: structure, process, and acquisition. *Canadian Journal of Psychology*, 39, 338-366.

Gonzales, E.G., & Kolers, P.A. (1982). Mental manipulation of arithmetic symbols. *Journal of Experimental Psychology: Learning, Memory, and Cognition,* 8, 308-319.

Groen, G.J., & Parkman, J.M. (1972). A Chronometric analysis of simple addition. *Psychological Review,* 79, 329-343.

Krueger, L.E. (1986). Why 2 x 2 = 5 looks so wrong: on the odd-even rule in product verification. *Memory & Cognition,* 14, 141-149.

Krueger, L.E., & Hallford, E.W. (1984). Why 2 + 2 = 5 looks so wrong: on the odd-even rule in sum verification. *Memory & Cognition,* 12, 171-180.

LeFevre, J., Bisanz, J., & MrKonjic, L. (1988). Cognitive arithmetic: evidence for obligatory activation of arithmetic fact. *Memory & Cognition,* 16, 45-53,

Parkman, J.M. (1972). Temporal aspects of simple multiplication and comparison. *Journal of Experimental Psychology*, 93, 437-444.

Parkman, J.M., & Groen, G.J. (1971). Temporal aspects of simple addition and comparison. *Journal of Experimental Psychology*, 89, 335-342.

Restle, F. (1970). Speed of adding and comparing numbers. *Journal of Experimental Psychology*, 83, 274-278.

Solso, R.L. (1991). *Cognitive Psychology*. Boston: Allyn and Cacon, Inc.

Stazyk, E.H., Ashcraft, M.H., & Hamann, M.S. (1982). A network approach to mental multiplication. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 8, 320-335.

Winkelman, J.H., & Schmidt, J. (1974). Associative confusions in mental arithmetic. *Journal of Experimental Psychology*, 102, 734-736.

Zbrodoff, N.J., & Logan, G.D. (1986). On the autonomy of mental processes: a case study of arithmetic. *Journal of Experimental Psychology: General*, 115, 118-130.