ON FINITE CONVERGENCE INDEX

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ABSTRACT

Chaatit, Mascioni, and Rosenthal defined the class of functions of finite Baire index and proved that the class forms an algebra and a lattice. Following that idea, in this paper we define $\mu((f_n))$, the finite convergence index of a given sequence of real-valued functions (f_n). Let $(f_n),(g_n)$ be sequences of real-valued functions on a Polish space X and (h_n) be any of the sequences $(f_n)+(g_n)$, $(f_n).(g_n)$, $\max\{(f_n),(g_n)\}$, $\min\{(f_n),(g_n)\}$, then we prove that $\mu((h_n)) \leq \mu((f_n)) + \mu((g_n))$.

Key words and Phrases: *finite convergence index, sequences of functions*

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1. INTRODUCTION

Throughout, let X be a Polish space, that is, a separable completely metrizable space. Chaatit, Mascioni and Rosenthal (1996) defined an index $i(f)$ of a bounded function that called finite Baire index. Previously, Kechris and Louveau (1990) and Haydon, Odell and Rosenthal (1991) have defined transfinite oscillation index $\beta(f)$ for a given function $f : X \to \mathbb{R}$. After that, several authors have studied the oscillation index for Baire-1 functions. (See, e.g.,Leung and Tang (2003), (2006)). In the case $f : X \to \mathbb{R}$ is a function of finite Baire index, it was proved by Chaatit, Mascioni and Rosenthal (1996) that $\beta(f) = i(f) + 1$.

Let $f, g : X \to \mathbb{R}$ be functions of finite Baire index, it was proved by Chaatit, Mascioni and Rosenthal (1996) that $i(h) \leq i(f) + i(g)$, where h is any of the functions $f + g, fg, \max\{f, g\}, \min\{f, g\}$. In this paper, following the idea in Chaatit, Mascioni and Rosenthal (1996), we define finite convergence index for a given sequence of real-valued functions (f_n) and prove similar results as above. We prove the following results : Let (f_n) and (g_n) be sequences of real-valued functions on X with finite convergence index. If (h_n) be any of the sequences $(f_n) + (g_n)$, $(f_n).(g_n)$, $\max\{(f_n),(g_n)\}$, $\min\{(f_n),(g_n)\}$, then

$$\mu((h_n)) \leq \mu((f_n)) + \mu((g_n))$$

where $\mu((h_n)), \mu((f_n))$ and $\mu((g_n))$ denote the finite convergence index of $(h_n), (f_n)$ and (g_n) respectively. These results do not assume the boundedness of the sequences involved unless for product case.

We begin by recalling the definition of the derivation introduced by Kechris and Louveau (1990). Here, we use $\mathbb{N}$ to denote the set of all non-negative integer numbers, that is, $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Suppose that (f_n) is a sequence of real-valued functions on a Polish space X. For any closed subset H of X and $\varepsilon > 0$, let $\mathcal{D}((f_n), \varepsilon, H)$ be the set of all $x \in H$ such that for every neighborhood U of x and any natural number m, there are two integers n_1, n_2 with $n_1 > n_2 > m$ and $y \in U \cap H$ such that $|f_{n_1}(y) - f_{n_2}(y)| \geq \varepsilon$. Now, we are ready to define $\mathcal{D}_j((f_n), \varepsilon, H)$ for any $j \in \mathbb{N}$ inductively. Let $\mathcal{D}_0((f_n)), \varepsilon, H) = H$. Suppose that $\mathcal{D}_j((f_n), \varepsilon, H)$ has been defined for some $j \in \mathbb{N}$, let $\mathcal{D}_{j+1}((f_n), \varepsilon, H) = \mathcal{D}((f_n), \varepsilon, \mathcal{D}_j((f_n), \varepsilon, H))$. It is not hard to see that $\mathcal{D}_j((f_n), \varepsilon, H)$ is closed for any $j \in \mathbb{N}$. The following fact was proved by Kechris and Louveau (1990).

Lemma 1. *Let (f_n) be a sequence of real-valued functions on X. Then for any closed subsets H, K of X and $\varepsilon > 0$,*

$$\mathcal{D}((f_n), \varepsilon, H \cup K) \subseteq \mathcal{D}((f_n), \varepsilon, H) \cup \mathcal{D}((f_n), \varepsilon, K).$$

Now, we are ready to define finite convergence index of the sequence (f_n).

Definition 2. *Let (f_n) be a sequence of real-valued functions on X and $H \subseteq X$ be a closed set. For $\varepsilon > 0$, we define $\mu_H((f_n), \varepsilon)$, the ε -convergence index of (f_n) on H, to be* $\sup\{j : \mathcal{D}_j((f_n), \varepsilon, H) \neq \emptyset\}$.

If there is no ambiguity about the space H, we write $\mu((f_n), \varepsilon)$ instead of $\mu_H((f_n), \varepsilon)$. The $\sup\{j : \mathcal{D}_j((f_n), \varepsilon, H) \neq \emptyset\}$ in Definition 2 is allowed to be ∞ by considering $\{j : \mathcal{D}_j((f_n), \varepsilon, H) \neq \emptyset\}$ as a subset of $\overline{\mathbb{R}}$, the extended real numbers. Thus,

$$\sup\{j : \mathcal{D}_j((f_n), \varepsilon, H) \neq \emptyset\} = \infty,$$

whenever the set $\{j : \mathcal{D}_j((f_n), \varepsilon, H) \neq \emptyset\}$ is unbounded.

There exist sequences of real-valued functions (f_n) on $H \subseteq X$ which we can find $j \in \mathbb{N}$ such that $\mathcal{D}_j((f_n), \varepsilon, H) = \emptyset$ for all $\varepsilon > 0$. For example, a sequence (f_n) which converges uniformly to a function f. It is not hard to see that for this uniform convergent sequence (f_n), we have $\mathcal{D}_1((f_n), \varepsilon, H) = \emptyset$ for all $\varepsilon > 0$. For this kind of sequence which there exists $j \in \mathbb{N}$ with $\mathcal{D}_j((f_n), \varepsilon, H) = \emptyset$ for all $\varepsilon > 0$, it is clear that $\{\mu((f_n), \varepsilon) : \varepsilon > 0\}$ is finite. We call such a sequence by a sequence of finite convergence index. We present the precisely definition in the following:

Definition 3. *A sequence (f_n) of real-valued functions on X is said to be of finite convergence index if there exists $j \in \mathbb{N}$ with $\mathcal{D}_j((f_n), \varepsilon, H) = \emptyset$ for all $\varepsilon > 0$. We then define $\mu((f_n))$, the convergence index of (f_n), by*

$$\mu((f_n)) = \max_{\varepsilon > 0} \mu((f_n), \varepsilon).$$

Clearly that if (f_n) is a uniform convergent sequence, then $\mu((f_n)) = 0$.

2. MAIN RESULTS

We prove the results by the method used in Chaatit, Mascioni and Rosenthal (1996) Theorem 1.3.

Theorem 4. *Let (f_n) and (g_n) be sequences of real-valued functions on X with finite convergence index. Then for any $\varepsilon > 0$, we have*

$$\mu((f_n) + (g_n), \varepsilon) \leq \mu((f_n), \frac{\varepsilon}{2}) + \mu((g_n), \frac{\varepsilon}{2}).$$

Proof. Let $\varepsilon > 0$ be given. First, for any closed subset H of X, we claim that

$$\mathcal{D}((f_n) + (g_n), \varepsilon, H) \subseteq \mathcal{D}((f_n), \frac{\varepsilon}{2}, H) \cup \mathcal{D}((g_n), \frac{\varepsilon}{2}, H). \quad (1)$$

Indeed, if $x \in H \backslash (\mathcal{D}((f_n), \frac{\varepsilon}{2}, H) \cup \mathcal{D}((g_n), \frac{\varepsilon}{2}, H))$, then there exists a neighborhood U of x and $n_0 \in \mathbb{N}$, such that for all integers $n > m > n_0$ and $y \in U \cap H$, we have

$$|f_n(y) - f_m(y)| < \frac{\varepsilon}{2} \text{ and } |g_n(y) - g_m(y)| < \frac{\varepsilon}{2}.$$

Therefore, for all $y \in U \cap H$ and $n > m > n_0$,

$$|f_n(y) + g_n(y) - (f_m(y) + g_m(y))| < \varepsilon,$$

which means $x \in H \setminus \mathcal{D}((f_n) + (g_n), \varepsilon, H)$.

For each $k \in \mathbb{N}$, and $\theta \in \{0,1\}^k = \{(t_1, t_2, \ldots, t_k) : t_i = 0 \textit{ or } 1, \ 1 \leq i \leq k\}$ we define subsets X_θ of X as follows :

$$X_0 = \mathcal{D}((f_n), \frac{\varepsilon}{2}, X), \ X_1 = \mathcal{D}((g_n), \frac{\varepsilon}{2}, X). \quad (2)$$

If $k \geq 1$ and $X_\theta = X_{(t_1, t_2, \ldots, t_k)}$ has been defined, let

$$X_{(t_1, \ldots, t_{k+1})} = \begin{cases} \mathcal{D}((f_n), \frac{\varepsilon}{2}, X_\theta) & \text{if } t_{k+1} = 0, \\ \mathcal{D}((g_n), \frac{\varepsilon}{2}, X_\theta) & \text{if } t_{k+1} = 1. \end{cases} \quad (3)$$

For all $k \in \mathbb{N}$, we claim that

$$\mathcal{D}_k((f_n) + (g_n), \varepsilon, X) \subseteq \bigcup_{\theta \in \{0,1\}^k} X_\theta. \quad (4)$$

We prove the claim by induction on k. For $k = 1$, this is just (1). Suppose that (4) is true for some $k \in \mathbb{N}$, then by Lemma 1 we have

$$\begin{aligned} &\mathcal{D}_{k+1}((f_n) + (g_n), \varepsilon, X) \\ &\quad = \mathcal{D}((f_n) + (g_n), \varepsilon, \mathcal{D}_k((f_n) + (g_n), \varepsilon, X)) \\ &\quad \subseteq \mathcal{D}((f_n) + (g_n), \varepsilon, \bigcup_{\theta \in \{0,1\}^k} X_\theta) \\ &\quad \subseteq \bigcup_{\theta \in \{0,1\}^k} \mathcal{D}((f_n) + (g_n), \varepsilon, X_\theta) \end{aligned}$$

$$\begin{aligned} &\subseteq \bigcup_{\theta\in\{0,1\}^k} \mathcal{D}((f_n),\varepsilon,X) \bigcup \\ &\quad \bigcup_{\theta\in\{0,1\}^k} \mathcal{D}((g_n),\varepsilon,X) \\ &= \bigcup\{X_{(t_1,\ldots,t_{k+1})} : \\ &\qquad (t_1,\ldots,t_{k+1}) \in \{0,1\}^{k+1}\} \end{aligned}$$

This finish the proof of (4).

Next, fix $k \in \mathbb{N}$ and $\theta = (t_1,\ldots,t_k) \in \{0,1\}^k$. Let

$$p(\theta) = \mathrm{card}\{1 \leq \mathrm{i} \leq \mathrm{k} : \mathrm{t_i} = 0\}$$

and

$$q(\theta) = \mathrm{card}\{1 \leq \mathrm{i} \leq \mathrm{k} : \mathrm{t_i} = 1\}$$

where card A denote cardinal number of a set A. We claim that

$$X_\theta \subseteq \mathcal{D}_{p(\theta)}((f_n),\frac{\varepsilon}{2},X) \cap \mathcal{D}_{q(\theta)}((g_n),\frac{\varepsilon}{2},X) \tag{5}$$

We prove (5) by induction on k. It follows from the definition of X_θ that (5) is true for $k=1$. Suppose that (5) holds for some $k \in \mathbb{N}$ and $(t_1,\ldots,t_{k+1})$ is given. Let $p = p(t_1,\ldots,t_k)$ and $q = q(t_1,\ldots,t_k)$. If $t_{k+1} = 1$, then $p(t_1,\ldots,t_{k+1}) = p$ and $q(t_1,\ldots,t_{k+1}) = q+1$. By assumption induction, we have

$$X_{(t_1,\ldots,t_{k+1})} \subseteq X_{(t_1,\ldots,t_k)} \subseteq \mathcal{D}_p((f_n),\frac{\varepsilon}{2},X)$$

and

$$\begin{aligned} X_{(t_1,\ldots,t_{k+1})} &= \mathcal{D}((g_n),\frac{\varepsilon}{2},X_{(t_1,\ldots,t_k)}) \\ &\subseteq \mathcal{D}((g_n),\frac{\varepsilon}{2},\mathcal{D}_q((g_n),\frac{\varepsilon}{2},X)) \\ &= \mathcal{D}_{q+1}((g_n),\frac{\varepsilon}{2},X). \end{aligned}$$

If $t_{k+1} = 0$, then $p(t_1,\ldots,t_{k+1}) = p+1$ and $q(t_1\ldots,t_{k+1}) = q$. Using similar reason, we have

$$X_{(t_1,\ldots,t_{k+1})} \subseteq \mathcal{D}_{p+1}((f_n),\frac{\varepsilon}{2},X) \cup \mathcal{D}_q((g_n),\frac{\varepsilon}{2},X).$$

Thus, (5) is proved for $k+1$.

Now, suppose that $\mathcal{D}_k((f_n)+(g_n),\varepsilon,X) \neq \emptyset$ for a given $k \in \mathbb{N}$. By (4) there exists $\theta = (t_1,\ldots,t_k) \in \{0,1\}^k$ such that $X_\theta \neq \emptyset$. Let $p(\theta) = \mathrm{card}\{1 \leq \mathrm{i} \leq \mathrm{k} : \mathrm{t_i} = 0\}$ and $q(\theta) = \mathrm{card}\{1 \leq \mathrm{i} \leq \mathrm{k} : \mathrm{t_i} = 1\}$, then by (5) we get

$$\mathcal{D}_{p(\theta)}((f_n),\frac{\varepsilon}{2},X) \neq \emptyset$$

and $\mathcal{D}_{q(\theta)}((g_n),\frac{\varepsilon}{2},X) \neq \emptyset$. Therefore,

$$k = p(\theta) + q(\theta) \leq \mu((f_n),\frac{\varepsilon}{2}) + \mu((g_n),\frac{\varepsilon}{2}).$$

This completes the proof.

The following corollary is obtained immediately from Theorem 4.

Corollary 5. *Let (f_n) and (g_n) be sequences of real-valued functions on X with finite convergence index. Then*

$$\mu((f_n)+(g_n)) \leq \mu((f_n)) + \mu((g_n)).$$

For next results, we use the usual meaning of product, minimum and maximum of two sequences of functions. More precisely, if (f_n) and (g_n) are sequences of real functions, we define product as $(f_n).(g_n) = (f_n.g_n)$. Minimum and maximum of (f_n) and (g_n) we denote by $(f_n)\wedge(g_n)$ and $(f_n)\vee(g_n)$ respectively, which is defined by $(f_n)\wedge(g_n) = (f_n\wedge g_n)$ and $(f_n)\vee(g_n) = (f_n \vee g_n)$. For a real function f, we denote $||f||_\infty = \sup\{|f(x)| : x \in X\}$.

Theorem 6. *Let (f_n) and (g_n) be sequences of real-valued functions on X with finite convergence index such that $\sup_n ||f_n||_\infty < \infty$ and $\sup_n ||g_n||_\infty < \infty$. Then*

$$\mu((f_n).(g_n)) \leq \mu((f_n)) + \mu((g_n)).$$

Proof. Let M be a number such that $\sup_n ||f_n||_\infty \leq M$ and $\sup_n ||g_n||_\infty \leq M$. It is enough to prove that that for every $\varepsilon > 0$,

$$\mu((f_n).(g_n),\varepsilon) \leq \mu((f_n),\frac{\varepsilon}{2M}) + \mu((g_n),\frac{\varepsilon}{2M}).$$

First, for any closed subset H of X, we claim that

$$\begin{aligned} \mathcal{D}((f_n).(g_n),\varepsilon,H) \subseteq\ & [\mathcal{D}((f_n),\frac{\varepsilon}{2M},H) \\ & \cup \mathcal{D}((g_n),\frac{\varepsilon}{2M},H)]. \end{aligned}$$

Indeed, if

$$x \in H \setminus (\mathcal{D}((f_n),\frac{\varepsilon}{2M},H) \cup \mathcal{D}((g_n),\frac{\varepsilon}{2M},H)),$$

then there exists a neighborhood U of x and $n_0 \in \mathbb{N}$ such that for all integers $n > m > n_0$

and all $y \in U \cap H$, we have

$$|f_n(y) - f_m(y)| < \frac{\varepsilon}{2M}$$

and

$$|g_n(y) - g_m(y)| < \frac{\varepsilon}{2M}.$$

Therefore, if $y \in U \cap H$ and $n > m > n_0$ we have

$$\begin{aligned} &|f_n.g_n(y) - f_m.g_m(y)| \\ &\quad \leq \quad M|g_n(y) - g_m(y)| + M|f_n(y) - f_m(y)| \\ &\quad < \quad \varepsilon. \end{aligned}$$

Thus, $x \in \mathcal{D}((f_n).(g_n), \varepsilon, H)$. Then we proceed exactly as in the proof of Theorem 4, except that the sets $X_{(t_1,\ldots,t_n)}$ are defined by replacing "$\frac{\varepsilon}{2}$" in (2) and (3) by "$\frac{\varepsilon}{2M}$".

Theorem 7. *Let (f_n) and (g_n) be sequences of real-valued functions on X with finite convergence index. Then*

$$\mu((f_n) \wedge (g_n)) \leq \mu((f_n)) + \mu((g_n))$$

and

$$\mu((f_n) \vee (g_n)) \leq \mu((f_n)) + \mu((g_n)).$$

Proof. Let $(h_n) = (f_n) \wedge (g_n)$. Again, to prove the first part of the theorem, it is enough to prove that for any $\varepsilon > 0$, we have

$$\mu((h_n), \varepsilon) \leq \mu((f_n), \varepsilon) + \mu((g_n), \varepsilon).$$

We claim that for any closed subsets H of X and $\varepsilon > 0$, we have

$$\mathcal{D}((h_n), \varepsilon, H) \subseteq \mathcal{D}(f_n), \varepsilon, H) \cup \mathcal{D}((g_n), \varepsilon, H).$$

To prove the claim, let $x \in \mathcal{D}((h_n), \varepsilon, H)$. For any neighborhood U of x and $k \in \mathbb{N}$, there are integers $n > m > k$ and $y \in U \cap H$ such that $|f_n(y) - f_m(y)| \geq \varepsilon$. There are two cases, that are, $|h_n(y) - h_m(y)| = h_n(y) - h_m(y)$ and $|h_n(y) - h_m(y)| = h_m(y) - h_n(y)$.

For the first case, if $h_m(y) = f_m(y)$, then

$$\begin{aligned} \varepsilon \quad &\leq \quad |h_n(y) - h_m(y)| \quad = \quad h_n(y) - f_m(y) \\ &\leq \quad f_n(y) - f_m(y) \quad = \quad |f_n(y) - f_m(y)| \end{aligned}$$

which implies that $x \in \mathcal{D}((f_n), \varepsilon, H)$. Similarly, if $h_m(y) = g_m(y)$, then

$$\begin{aligned} \varepsilon \quad &\leq \quad |h_n(y) - h_m(y)| \quad = \quad h_n(y) - g_m(y) \\ &\leq \quad g_n(y) - g_m(y) \quad = \quad |g_n(y) - g_m(y)| \end{aligned}$$

which implies that $x \in \mathcal{D}((g_n), \varepsilon, H)$.

For the second case, we can show in the same way. To finish the proof, then we proceed exactly as in the proof of Theorem 4, except that the sets $X_{(t_1,\ldots,t_n)}$ are defined by replacing "$\frac{\varepsilon}{2}$ " in (2) and (3) by "ε".

The second assertion in the theorem, that is, $\mu((f_n) \vee (g_n)) \leq \mu((f_n)) + \mu((g_n))$ can be proved by similar technique.

3. CONCLUDING REMARKS

Chaatit, Mascioni and Rosenthal (1996) use finite Baire index to characterize the set of all uniform limits of functions in the class of differences of bounded semicontinuous functions. We hope that finite convergence index can be used to characterize a class of functions.

REFERENCES

Chaatit, F., Mascioni, V., and Rosenthal, H.P., 1996, On functions of finite Baire index, *J. Funct. Anal.*, **142**, 277–295.

Haydon, R., Odell, E., and Rosenthal, H.P, 1991, On certain classes of Baire-1 functions with applications to Banach space theory, *in : Functional Analysis Lecture Notes in Math.*1470, Springer, New York, 1-35.

Kechris, A.S. and Louveau, A., 1990, A classification of Baire class 1 functions, *Trans. Amer. Math. Soc.*, **318**, 209–326.

Leung, D.H and Tang, W.-K., 2003, Functions of Baire class one, *Fund. Math.*, **179**, 225–247.

Leung, D.H. and Tang, W.-K., 2006, Extension of functions with small oscillation, *Fund. Math.*, **192**, 183–193.