Computing Greeks by Finite Difference using Monte Carlo Simulation and Variance Reduction Techniques

Komputasi Greeks dengan Metode Finite Difference menggunakan Simulasi Monte Carlo dan Teknik Reduksi Variansi

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Abstract

This paper looks at the use of the Monte Carlo method for computing Greeks. The algorithms of the finite difference method for estimating Greeks using Monte Carlo simulation and variance reduction techniques (i.e. common random numbers and antithetic variates) are presented. The Black-Scholes model is used as a benchmark model to analyse and test the numerical methods. It is shown that the number of simulations affects the performance the most, and there are some techniques to reduce the error. In the case of an option with discontinuous payoff, the method does not work quite well when the current time is approaching the maturity time.

***Keywords**: Options, Greeks, Monte Carlo Method, Finite Difference Method, and Black-Scholes Model.*

Abstrak

Makalah ini membahas penggunaan metode Monte Carlo untuk komputasi Greeks. Algoritma metode beda hingga untuk memperkirakan Greeks menggunakan teknik simulasi Monte Carlo dan teknik reduksi varians (yaitu nomor acak umum dan varians antitetis) disajikan. Model Black-Scholes digunakan sebagai model acuan untuk menganalisa dan menguji metode numerik. Hal ini menunjukkan bahwa jumlah simulasi mempengaruhi kinerja yang paling banyak, dan ada beberapa teknik untuk mengurangi kesalahan. Dalam kasus options dengan payoff diskontinu, metode ini tidak berjalan dengan baik saat waktu sekarang mendekati waktu jatuh tempo.

***Kata kunci**: Options, Greeks, Metode Monte Carlo, Metode Beda Hingga, dan Model Black-Scholes*

1. Introduction

Any investment in financial markets can be a highly effective way to gain returns. However, it is also important to be aware that all investments carry some risks due to factors such as inflation and economic downturns. The reduction of risk has been recognized as vital in the investment strategy after Modern Portfolio Theory was introduced in 1952 through a work entitled ”Portfolio Selection” by Harry Markowitz. One form of risk reduction, also called hedging, is to diversify the investment into assets that are not strongly correlated to each other, such as bonds versus stocks, or stocks in food versus financial industry.

In the last four decades, one type of financial assets called derivatives have become increasingly important and popular in finance so that in many cases they even exceed the value of the markets of the underlying assets (Hull, 2012). A derivative, such as options, essentially is a contract which its value depends on the value of other financial assets like stocks or commodities. There are some payoffs that can not be achieved without derivatives, or can only achieve at greater cost. The main benefit of derivatives is a permission for investors to hedge risks that otherwise would not be possible to hedge.

In order to manage the risk associated with option trading, quite sophisticated hedging procedures are required. These procedures involve calculating and monitoring a set of quantities known as the option sensitivities, more commonly referred to as the Greeks. Below are the lists of most common Greeks (see e.g. Lyuu, 2001; Haug, 2007; Hull, 2012):

1. Delta reflects the sensitivity of the value of the option to changes in the price of the underlying asset.
2. Gamma is the rate of change of delta with respect to the price of the underlying asset.
3. Vega is a measure of the sensitivity of of the value of the option to changes in the volatility of the price of the underlying asset.
4. Theta is a measure of the value of the option would be expected to change to reflect the passage of time.
5. Rho is a measure of the value of the option would change for an incremental move in short-term interest rates.

The Greeks are vital measurements in risk management for option traders (Higham, 2004). For instance, suppose that a delta of an option on a stock is 0.4, the stock price changes by a small amount, and no other pricing variables change. In theory, the option price changes by about 40% of that amount. It is the short position of the trader in order to create a risk-less portfolio. The construction of a risk-less portfolio is referred to as delta hedging. Delta hedging is essential for an option trader to ensure that the overall delta of their position is close to zero so that changes in the underlying assets do not affect the overall value of his position.

It is important to remember that the enhanced (i.e. hedged) portfolio has to be adjusted over each time period since the delta of an option does not remain constant. This procedure, where the hedge is adjusted on a regular basis, is known as dynamic hedging. Rebalancing the portfolios can be very expensive, so that most option traders concentrate on assessing risk instead of eliminating risk (Straja, 2010). This means that we do not only need to be able to value the option, but also to calculate the Greeks, which are used to quantify the different aspects of risk.

Numerical computation provides a powerful means to compute a variety of purposes in cases where no analytical formulas are available. It is also important to consider models, which have higher complexity and fewer assumptions, as the case that happens in the real-world. The analytical approach gives the exact results, yet it requires restrictive assumptions to make the problem tractable. Therefore, sometimes the numerical approach is more flexible, particularly in the scope for developing the model (McLeish, 2005).

Market models have been developed to provide a foundation for derivatives valuation and determination of hedging strategies. The prices of the underlying assets in an efficient market are often modelled as a stochastic process. Then, using the assumption of no arbitrage and establishing the risk-neutral measure, valuing generic derivatives can be represented as an expected value (see e.g. Kwok, 1998; Bjork, 2003). Thus, valuing derivatives reduces to calculating expectations. Rewriting the relevant expectation as an integral of the random variable with respect to its probability measure, we would often find the integration in very large dimensions (Glasserman, 2004). This is the case in which Monte Carlo method becomes increasingly attractive since the error of the method is independent of the dimension.

The Monte Carlo method compute the estimates' expected value by performing a random sampling of a certain random variable (Atanassov and Dimov, 2007). The expected value is a function of the solution to a stochastic differential equation (SDE). Boyle (1977) first proposed the application of Monte Carlo method to evaluate the value of European options. For an overview of Monte Carlo methods for option pricing see, for example, Boyle, Broadie, and Glasserman (1997) and the improvement of Monte Carlo error through variance reduction techniques have been discussed in e.g. Glasserman (2004) and Asmussen and Glynn (2007). For a more comprehensive analysis of numerical methods for solving SDEs can be found in e.g. Kloeden and Platen (1999) and Milstein and Tretyakov (2004).

The Greeks of the options, in which payoff only depends on final time or has an expected value explicitly (e.g. simple European options), have formulas derived from the Black-Scholes (1973) formula. However, there are a lot more types of options, e.g. European-type Asian options and American options, in which no analytical solution is given. We need to resort to the numerical differentiation in order to calculate the Greeks for those types of options and other models, such as Heston (1993) model. The main idea of the numerical differentiation is to employ the finite difference method as the approximation. The finite difference method together with the Monte Carlo method can be employed to compute Greeks. As a result, there are three errors, which are: the error of the approximation from finite difference method, the statistical error of the Monte Carlo method and the error of numerical integration.

As discussed in Glynn (1989), the forward finite difference method has the convergence rate not higher than $M^{-1/4}$ where M is the sample size. One way to improve the convergence rate is by using a central finite difference approximation to the derivative and obtains the improved convergence rate $M^{-1/3}$. Furthermore, by using *common random numbers* for the Monte Carlo estimators, one can achieve $M^{-1/2}$ which is the best possible rate for a Monte Carlo method, see e.g. Glasserman and Yao (1992).

This paper looks at numerical methods for accurately computing the Greeks which have no analytical solution. The finite difference method for estimating the Greeks using Monte Carlo simulation and the variance reduction techniques (i.e. *common random numbers* and *antithetic variates*) are presented. In order to check the accuracy of the methods, we present numerical results using the Black-Scholes model in which price and Greeks of the European options are available by the Black-Scholes exact formula. The main objective of this paper is to apply, analyse and test the Monte Carlo method for computing Greeks.

2. Computing Option Prices and Greeks

The derivatives of the simple European option value in the Black-Scholes model are given by closed form formulas. From a numerical point of view, they are easy to compute. We only have to compute the distribution function of a $N(0,1)$. However, in the case where we do not have closed form formulas, prices and Greeks must be calculated by numerical approximations. For the Greeks, we will restrict ourselves to delta (Δ), theta (Θ), and rho (ρ).

In this section, firstly, we briey summarize the pricing of options with the Monte Carlo simulation based on (Glasserman, 2004). Then, we discuss the numerical methods for computing Greeks by using finite difference and the Monte Carlo method. Finally, we run some simulations to apply, analyze, and test the methods.

To price the option, we model the dynamics of the underlying assets under the risk-neutral measure. It is easier to produce sample paths with a risk-free rate r under risk-neutral $\mathbb{Q}$ measure since under the real probability measure $\mathbb{Q}$, the drift parameter μ associated with varying risk preferences of investors is much harder to estimate, while r can be estimated as a risk-less interest rate.

2.1 Pricing Option with Monte Carlo Simulation

Consider a stochastic process to model the stock price S_t at time t. Assume the option expires at time $T$and $0 \leq t_0 \leq T$. By dividing the interval $[t_0, T]$ into N equal parts of length $h = T/N$, a simulated path of S_t is given by $\boldsymbol{S} = (S_{t_0}, S_{t_1}, S_{t_2}, \dots, S_{t_N})$. Using geometric Brownian motion to model the stock price S_t , the solution to the dynamics under the risk neutral probability measure ($\mu = r$):

$$S_t = x_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma B_t\right). \quad (1)$$

The simulated path of S_t is given by $\mathbf{S} = (S_0, S_1, S_2, \ldots, S_N))$. Each element in $\mathbf{S}$ is given by the recursive formula

$$\begin{aligned} S_i &= S_{i-1} \exp\left((r - \frac{1}{2}\sigma^2)h + \sigma\sqrt{h}z_i \right), \quad 1 \le i \le N \\ S_0 &= x_0, \end{aligned} \tag{2}$$

for the vector $\boldsymbol{z} = (z_1, z_2, \ldots, z_N)$ where $z_i \sim N(0,1)$. Figure 1 shows four paths of this stock price process.

We denote the payoff function for an option by $f(\cdot)$, the strike price by K, and the current time by t. The price of the option is then given by

$$\Pi = E_Q[e^{-r(T-t)} f(\mathbf{S}, K)]. \tag{3}$$

To evaluate the expectation, we simulate paths of the underlying assets over the time interval, according to their risk-neutral dynamics. We calculate the discounted payoff $e^{-r(T-t)} f(\mathbf{S}, K)$ on each path, then the average over the paths is our estimate of the optio price.

The number of paths is chosen to be a suitably large number to yield more precise approximations to the option price. Each simulation of S differs only through the vector $\boldsymbol{z}$, so for the M simulations: $\mathbf{S}^j = \mathbf{S}(\mathbf{z}^j; x, r, \sigma, T, t, N), 1 \le j \le M$, where $\boldsymbol{z}$ is a vector of random variables, x is the initial stock price, r is the risk interest rate, σ is the volatility, T is maturity time, and t is the current time. Then, the Monte Carlo approximation of the option price in equation (3) is given as follows:

$$\Pi_M = \frac{1}{M} e^{-r(T-t)} \sum_{j=1}^{M} f\left(\mathbf{S}(\mathbf{z}^j; x, r, \sigma, T, t, N), K\right). \tag{4}$$

2.2 Numerical Methods For Computing Greeks

The Greeks then are given by

$$\begin{aligned} \Delta &= \partial_x E_Q[e^{-r(T-t)} f(\mathbf{S}, K)]; \\ \Theta &= -\partial_T E_Q[e^{-r(T-t)} f(\mathbf{S}, K)]; \\ \rho &= \partial_r E_Q[e^{-r(T-t)} f(\mathbf{S}, K)]. \end{aligned} \tag{5}$$

The combination of methods that is obvious to use is the Monte Carlo method for computing the prices and deducing a finite difference estimator for Greeks. In many cases, direct (without re-simulation) approachhes are possible. Let x be a parameter of interest. A Greek with respect to x equals $\partial_x E_Q[e^{-r(T-t)} f(\mathbf{S}, K)$. The direct method will depend on the validity of the interchange of the order of expectation and differentiation (Lyuu and Teng, 2011), i.e.

$$\partial_x E_Q[e^{-r(T-t)} f(\mathbf{S}(x), K)] = E_Q\left[\partial_x \left(e^{-r(T-t)} f(\mathbf{S}(x), K)\right)\right]. \tag{6}$$

In equation (6), the right-hand side equals the desired Greek. Broadie and Glasserman (1996) establish a set of conditions on the payoff function for equation (6) to hold. For example, the function must exists and be smooth enough by derivation of the payoff or by derivation of the transition probability density. We may approach the Greek by forward finite difference

$$\frac{1}{\Delta x}\left(E_Q[e^{-r(T-t)} f(\mathbf{S}(x + \Delta x), K)] - E_Q[e^{-r(T-t)} f(\mathbf{S}(x), K)]\right) \tag{7}$$

or, leading to better convergence properties, by central finite difference

$$\frac{1}{2\Delta x}\left(E_Q[e^{-r(T-t)}f(\mathbf{S}(x+\Delta x),K)] - E_Q[e^{-r(T-t)}f(\mathbf{S}(x-\Delta x),K)]\right). \tag{8}$$

If Δx is too large, the finite difference approximation error becomes significant, while if it is too small the variance can become very large if the payoff function is not differentiable (Giles, 2007). Therefore, care must be taken in the choice of Δx.

The expectations are estimated by Monte Carlo simulation. In order to optimize algorithms, the variance reduction technique of common random numbers should be used to estimate both expectations in equation (8). Thus, the Greeks in equation (5) can be estimated in these approaches by

$$\begin{aligned}
\hat{\Delta}_M &= \frac{1}{2M\Delta x}e^{-r(T-t)}\sum_{j=1}^{M} f\left(\mathbf{S}(\mathbf{z}^j; x+\Delta x, r, \sigma, T, t, N), K\right) && (9)\\
&\quad -e^{-r(T-t)} f\left(\mathbf{S}(\mathbf{z}^j; x-\Delta x, r, \sigma, T, t, N), K\right);\\
\hat{\Theta}_M &= \frac{-1}{2M\Delta t}\sum_{j=1}^{M} e^{-r(T+\Delta t-t)} f\left(\mathbf{S}(\mathbf{z}^j; x, r, \sigma, T+\Delta t, t, N), K\right) && (10)\\
&\quad -e^{-r(T-\Delta t-t)} f\left(\mathbf{S}(\mathbf{z}^j; x, r, \sigma, T-\Delta t, t, N), K\right);\\
\hat{\rho}_M &= \frac{1}{2M\Delta r}\sum_{j=1}^{M} e^{-(r+\Delta r)(T-t)} f\left(\mathbf{S}(\mathbf{z}^j; x, r+\Delta r, \sigma, T, t, N), K\right) && (11)\\
&\quad -e^{-(r-\Delta r)(T-t)} f\left(\mathbf{S}(\mathbf{z}^j; x, r-\Delta r, \sigma, T, t, N), K\right).
\end{aligned}$$

Now, we should remember that we are

a. approximating the solution to the SDE by numerical integration, which has $(\mathcal{O}(h^p))$ truncation error,
b. approximating the expectation by Monte Carlo method. This Monte Carlo error is of size $\left(\mathcal{O}\left(1/\sqrt{M}\right)\right)$ and
c. approximating the differentiation by central forward difference, which has error of $\mathcal{O}((\Delta x)^2)$.

These are thus the sources of the error in the numerical methods for computing Greeks using central finite difference and the Monte Carlo method. In the case of weak Euler approximation ($p = 1$) for the system of SDEs, Milstein and Tretyakov (2005) have shown the total error R of the numerical methods when payoff functions are sufficiently smooth,

$$R \sim \mathcal{O}\left((\Delta x)^2\right) + \mathcal{O}(h) + \mathcal{O}\left(\frac{h^2}{\Delta x}\right) + \mathcal{O}\left(\frac{1}{\sqrt{M}}\right); \tag{12}$$

if we put $\Delta x = \alpha h^{\beta}, \alpha > 0, 1/2 \leq \beta \leq 1$, then

$$R \sim \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{\sqrt{M}}\right). \tag{13}$$

3. Simulations and Results

In this section, we will implement numerical methods for computing the price and Greeks of the option as we have discussed above using Python language. We will show that the Monte Carlo approximations are close to the analytical values, investigate the impact of the number of paths, the size of timesteps, and the performances of variance reduction technique by Antithetic Variates in Monte Carlo calculations. Finally, we will investigate the methods for an option with discontinuous function. In order to check the accuracy, we will focus on the price and Greeks of European options which are available by the Black-Scholes exact formula.

We use a random-number generator available on Python language (Van Rossum, 1995). Random is the module that implements pseudorandom number generators for various distributions on this language, and specifically random.normal variate(*mu*, *sigma*) generate specific normal distribution with *mu* is the mean, and *sigma* is the standard deviation.

3.1 Implementing the Methods

We will compute the Monte Carlo approximations with corresponding Monte Carlo error for the price, delta, theta, and rho of the option. Consider an underlying process S that follows geometric Brownian motion in equation (1). Using the algorithm for simulating a gBm process, we provide a detailed algorithm for pricing a one-dimensional European call option with strike price K, maturity time T, and constant timesteps as follows:

1. Set up a discrete time framework using N timesteps of the equal size $h = T/N$, then each time point $t_i = ih$ for $i = 1, \ldots, N$. Assume M sample paths are used independently.
2. Generate $N \times M$ paths matrix $S_i^{(j)}$ and M payoff vector $X^{(j)}$ as follows:

$$\begin{aligned} S_0^{(j)} &= x_0, \\ S_i^{(j)} &= S_{i-1}^{(j)} e^{(r-\frac{1}{2}\sigma^2)h+\sigma\sqrt{h}z_i^{(j)}}, \\ X^{(j)} &= max(S_N^{(j)} - K, 0), \end{aligned} \tag{14}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ and where $z_i^{(j)}$ is an i.i.d. random variable arising from a normal distribution for j-the sample path.

3. Calculate the discounted payoffs at the risk free rate $\boldsymbol{DC^{(j)} = e^{-rt}X^{(j)}}$.
4. Average the discounted cash flows over M sample paths, that is,
5. Calculate the confidence interval with probability 95%.

For computing the Greeks, generally, the algorithm is not that much different from computing the price. We use another numerical method beside the Monte Carlo method, which is the approximation of the differentiation using the central finite difference approach. This requires Δx to be chosen. We can see from equation (12) that it is good to take Δx proportional to h so that equation (13) can be obtained. We use Δx as in Milstein and Tretyakov (2005), $\Delta x = \alpha h^{1/2}, \alpha > 0$ so that the error would be smaller than equation (12). We apply the common random numbers method in computing Greeks to minimize the variance. Equations (9), (10), and (11) are the algorithms of delta ($\widehat{\Delta}_M$), theta ($\widehat{\Theta}_M$), and rho ($\hat{\rho}_M$) respectively.

Variance reduction by antithetic variates (AV) is a method to increase the accuracy of the Monte Carlo method by doubling the sample size. For each $j = 1, \ldots, M$ use the sequence $\{z_1^{(j)}, \ldots, z_N^{(j)}\}$ in equation (2) to simulate a payoff $f(S^{j+}, K)$ and also use the sequence $\{-z_1^{(j)}, \ldots, -z_N^{(j)}\}$ in equation (2) to simulate an associated payoff $f(S^{j-}, K)$. Now the payoffs are simulated $(f(S^{j+}, K), f(S^{j-}, K))$. This is the program that we ran to evaluate the variance reduction in pricing an option.

Table 1. The set of parameters to apply and analyse the methods of computing price and Greeks.

Parameter		Value	Description
S	=	100	(initial asset value)
K	=	100	(strike price)
r	=	0.05	(risk-free interest rate)
σ	=	0.2	(volatility)
T	=	1	(maturity time)
t	=	0	(current time)

Table 1 contains a set of parameters that were used to apply and analyse the Monte Carlo methods of pricing and computing Greeks of an option. The favourable feature of the Monte Carlo method is the possibility to estimate the dominant error in most of its computations, namely the Monte Carlo error. The Monte Carlo error is a statistical confidence interval. In all simulations, we use the confidence interval with probability 0.95. The sign $\pm$ in all tables and descriptions in thissection only reflects the Monte Carlo error.

3.2 Number of paths

Using the set of parameters in Table 1, ($N = 10$), and ($M = 1$), we get $\Pi_M = 14.88980, \widehat{\Delta}_M = 1.0015, \widehat{\Theta}_M = -3.8324$, and $\hat{\rho}_M = 95.1625$. Certainly, there is no Monte Carlo error for one simulation since no variance occurred and the error of the Monte Carlo estimates seems to be very large. From a law of large numbers theorem we have that the average of the results obtained from resimulation in a large number of times converges almost surely to the expected value. Thus we increase the number of paths.

Table 2. Results of the Monte Carlo (MC) approximations of price, delta, theta, and rho of European call option under the parameter setting: $N = 100$ and $\alpha = 2$. The exact values are $\Pi = 10.4505, \Delta = 0.6368, \Theta = -6.4140$ and $\rho = 53.2324$.

M	Π_M	$\hat{\Delta}_M$	$\hat{\Theta}_M$	$\hat{\rho}_M$
10^2	8.2142 ± 2.4514	0.5577 ± 0.0443	-5.0762 ± 0.5845	46.1366 ± 2.5982
10^3	10.2119 ± 0.9021	0.6394 ± 0.0141	-6.3086 ± 0.2165	49.8812 ± 0.8438
10^4	10.5055 ± 0.2923	0.6321 ± 0.0045	-6.4668 ± 0.0707	50.3253 ± 0.2711
10^5	10.4856 ± 0.0913	0.6360 ± 0.0014	-6.4524 ± 0.0220	50.4854 ± 0.0855
10^6	10.4515 ± 0.0288	0.6365 ± 0.0004	-6.4353 ± 0.0069	50.4696 ± 0.0270

Table 2 shows the results with different numbers of simulations. We see that the Monte Carlo approximations get closer to the exact solution and the Monte Carlo errors get smaller as the number of simulated paths increases. More precisely, the convergence rate of the Monte Carlo error is $1/\sqrt{M}$ (see Figure 2). This confirms that the estimation error is ($\mathcal{O}(1/\sqrt{M})$).

The Monte Carlo error of the delta estimator is the smallest among those Greeks due to the variance of the delta estimator being the smallest. Theta estimator and rho estimator, have the difference increment in the discounted factors as well as in the payoff function (see equation (10) and (11)). However, equation (9) shows that the difference increment in the delta only affects the payoff and does not change the discounted factor. Thus, the variance, which implies the error, of the delta estimator is the smallest among them.

The Monte Carlo method does not provide the exact solution of the problem, but a confidence interval includes the solution with a given probability. We can see from Table 2 that the confidence interval with probability 0.95 of the estimate Π_M in each number of paths includes the analytical solution. However, it does not always work for the Greeks because of the bias of the Greeks estimators. The bias of the Greeks estimators has two parts: one due to timestepping (h) and the other is the discretization of the derivative (Δx).

Some of the exact solutions of delta and theta estimators in Table 2 are not in the corresponding confidence intervals, we do not even find the exact solution of rho in any of the corresponding confidence intervals. It might happen due to the selection of N which affects h and also Δx (since $\Delta x = \alpha h^{1/2}$). Under the set of parameter in Table 2, the size of the time-step is probably not small enough to estimate the rho whereas the rho estimator requires a very small size of time-step.

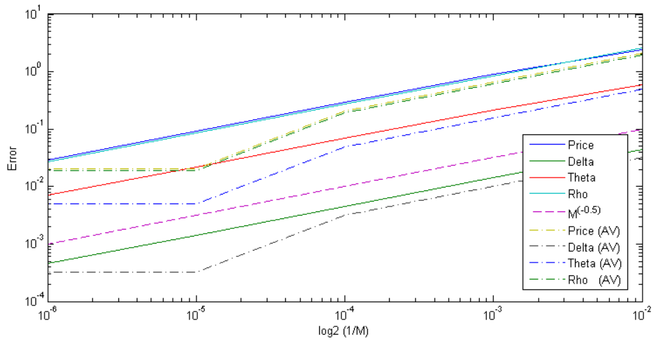

Figure 2. Log-log plot of number of paths versus the Monte Carlo error with and without Antithetic Variates (AV) under the parameter setting of Table 2 and Table 5

3.3 Size of time-steps

We simplify the bias of the estimators to be one part, which is the size of time-steps (h). Therefore, we prefer to choose Δx that is proportional to the h. In this section, we investigate the behaviour of the method as $h \to 0$. In other words, we apply the method in different numbers of time-steps (N) with $N \to \infty$. The finite difference method was employed to compute the Greeks as the numerical differentiation method in this study. This discretization error does not appear in pricing. As a result, there is an additional error of approximating the differentiation in computing the Greeks. We have discussed earlier in this section that care must be taken in the choice of the optimal difference increment (Δx) and we have chosen it as proportional to h, $\Delta x = \alpha h^{1/2}, \alpha > 0$.

We can estimate the weak error by comparing the Monte Carlo estimate with the exact solution. We reduce the Monte Carlo error by increasing the number of simulations so that the Monte Carlo can be neglected. Table 3 and Table 4 show the impact of h on the methods with $\alpha = 1$ and $\alpha = 2$ respectively. We choose some numbers of time-steps, i.e. $N = 4, 5, 8, 10, 20, 25, 40, 50, 80, 200$, and the size of timestep is $h = \frac{T-t}{N}$. Figure 3 presents the log-log plot error for the Euler method and the Monte Carlo method for each of the Greeks and helps us in interpreting Table 3 and Table 4.

Before we continue to look at the results, we summarize the relevant theories, the theory of the estimation of the total error in the valuation of Greeks. All the formulas given here relies heavily on Milstein and Tretyakov (2005). Consider the price of an option $u(t,x) = E[f(X_{t,x}(T))]$, where $f(X(T))$ is payoff function at maturity time T and $X_{t,x}(s)$ is the solution of stochastic differential equation. The Greeks of the option

$$\frac{\partial u}{\partial x}(t,x) = \frac{E[f(\bar{X}_{t,x+\alpha h^{1/2}}(T))] - E[f(\bar{X}_{t,x-\alpha h^{1/2}}(T))]}{2\alpha h^{1/2}} + \mathcal{O}(h) \tag{15}$$

by central finite difference method. The form $O(h)$ in equation (15) is the error due to the numerical differentiation. Then, both expectations in equation (15) are estimated by the Monte Carlo method and common random numbers technique,

$$\hat{\partial} u(t,x) = \frac{1}{2M\alpha h^{1/2}} \sum_{m=1}^{M} [f(\bar{X}^{(m)}_{t,x+\alpha h^{1/2}}(T)) - f(\bar{X}^{(m)}_{t,x-\alpha h^{1/2}}(T))]. \tag{16}$$

Table 3. Impact of h on the Monte Carlo error for computing the price and Greeks of European calls using the Monte Carlo method option under the parameter setting: $M = 10^6$ and $\alpha = 1$. The exact values are $\Delta = 0.6368$, $\Theta = -6.4140$ and $\rho = 53.2324$.

h	$\hat{\Delta}_M$	$\hat{\Theta}_M$	$\hat{\rho}_M$
0.25	0.6372 ± 0.0011	-6.5526 ± 0.0177	42.2160 ± 0.0388
0.2	0.6364 ± 0.0010	-6.5227 ± 0.0158	43.5981 ± 0.0382
0.125	0.6368 ± 0.0008	-6.4836 ± 0.0124	46.2209 ± 0.0360
0.1	0.6360 ± 0.0007	-6.4577 ± 0.0111	47.2667 ± 0.0345
0.05	0.6374 ± 0.0005	-6.4442 ± 0.0078	49.9048 ± 0.0289
0.04	0.6376 ± 0.0005	-6.4518 ± 0.0070	50.5389 ± 0.0270
0.025	0.6365 ± 0.0004	-6.4285 ± 0.0055	51.4257 ± 0.0230
0.02	0.6367 ± 0.0003	-6.4261 ± 0.0049	51.7785 ± 0.0211
0.0125	0.6372 ± 0.0003	-6.4215 ± 0.0039	52.3213 ± 0.0175
0.01	0.6364 ± 0.0002	-6.4106 ± 0.0035	52.4588 ± 0.0160

Table 4. Impact of h on the Monte Carlo error for computing the price and Greeks of European calls using the Monte Carlo method option under the parameter setting: $M = 10^6$ and $\alpha = 2$. The exact values are$\Delta = 0.6368$, $\Theta = -6.4140$ and $\rho = 53.2324$.

h	$\hat{\Delta}_M$	$\hat{\Theta}_M$	$\hat{\rho}_M$
0.25	0.6366 ± 0.0022	-8.0611 ± 0.0442	32.4985 ± 0.0396
0.2	0.6367 ± 0.0020	-7.1189 ± 0.0347	34.1669 ± 0.0396
0.125	0.6363 ± 0.0016	-6.7313 ± 0.0259	37.5262 ± 0.0396
0.1	0.6360 ± 0.0014	-6.6574 ± 0.0229	39.0800 ± 0.0395
0.05	0.6369 ± 0.0010	-6.5251 ± 0.0158	43.6015 ± 0.0382
0.04	0.6364 ± 0.0009	-6.4875 ± 0.0141	44.8711 ± 0.0372
0.025	0.6374 ± 0.0007	-6.4680 ± 0.0111	47.3174 ± 0.0345
0.02	0.6366 ± 0.0006	-6.4553 ± 0.0099	48.2335 ± 0.0329
0.0125	0.6369 ± 0.0005	-6.4459 ± 0.0078	49.8738 ± 0.0290
0.01	0.6366 ± 0.0005	-6.43533 ± 0.0070	50.4696 ± 0.0270

If we rewrite equation (27), we have

$$\frac{\partial u}{\partial x}(t,x) = \hat{\partial} u(t,x) + \frac{1}{2\alpha h^{1/2}} r_{\hat{\partial} u} + \mathcal{O}(h), \tag{17}$$

where $r_{\hat{\partial}_u}$ is the Monte Carlo error of computing $E[f(\bar{X}_{t,x+\alpha h^{1/2}}(T)) - f(\bar{X}_{t,x-\alpha h^{1/2}}(T))]$ in equation (16). The variance of the factor $r_{\hat{\partial}_u}$at the Monte Carlo error in equation (17),

$$Var(r_{\hat{\partial} u}) = \frac{Var[f(\bar{X}_{t,x+\alpha h^{1/2}}(T)) - f(\bar{X}_{t,x-\alpha h^{1/2}}(T))]}{M}. \tag{18}$$

since $Var\left[f\left(\bar{X}_{t,x+\alpha h^{1/2}}(T)\right) - f\left(\bar{X}_{t,x-\alpha h^{1/2}}(T)\right)\right] \leq Ch$, where C is positive constants independent of h, then the Monte Carlo error will be

$$r_{\hat{\partial} u} \sim \mathcal{O}\left(\sqrt{\frac{h}{M}}\right). \tag{19}$$

Thus the total error $R_{\hat{\partial}_u}$ in the evaluation of $\partial_u/\partial_x$from equation (28) is estimated as

$$R_{\hat{\partial} u} \sim \mathcal{O}(h) + \mathcal{O}\left(\frac{1}{\sqrt{M}}\right). \tag{20}$$

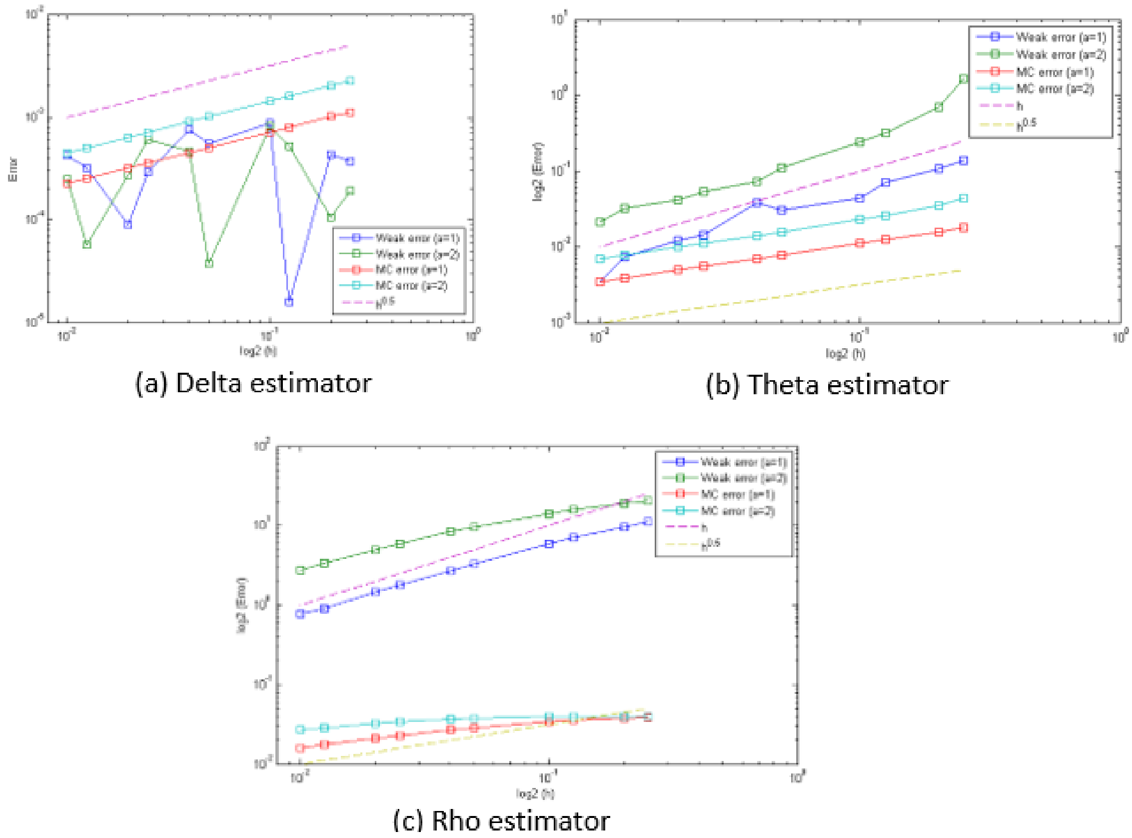

Figure 3. Log-log plot of size of time-step versus the error under the parameter setting of Table 3 and Table 4: (a) Delta estimator, (b) Theta estimator, and (c) Rho estimator

Firstly, we observe the results of the rho estimator. For both $\alpha = 1$ and $\alpha = 2$, the bias of the estimator has a weak order convergence rate $\lambda = 1$ as h get smaller. It starts to follow the convergence rate at $h = 0.05$ or $N = 20$. The Monte Carlo error of the rho estimator remains unchanged at the first four numbers of h. Yet, the error becomes proportional to $h^{1/2}$ as size of time-steps get smaller. Then, we get the error as in equation (19) from this experiment about size of time-steps ($\mathcal{O}(\sqrt{h})$) and previous experiment about number of paths ($\mathcal{O}(1/\sqrt{M})$). See Milstein and Tretyakov (2005) for the detail explanation about the total error in equation (20).

Figure 3 (b) shows that the convergence of theta estimator just occurs only in some timesteps even though the trend of the weak error is in line with the theory. It may have happened due to the method we use for evaluating theta. Milstein and Tretyakov mentioned that the direct application of the finite difference method is not so effective for evaluating theta from the computational point of view and pointed out another way for evaluating theta in Milstein and Schoenmakers (2002).

The delta estimator in Table 3 and Table 4 does not show a weak order convergence, it shows just a random error (see Figure 3 (a)). It may have happened because the number of simulations is not big enough to kill the Monte Carlo error. Thus, an alternative to showing the convergence of this estimator is to increase the number of simulated paths. However, it is too expensive and we do not have the computational power to check it. Another alternative to check the convergence is to set up another experiment with a set of parameters such that the problem gives a big error. The problem is usually related to a much longer maturity time.

3.4 Variance reduction by Antithetic Variables (AV)

We have seen that the standard Monte Carlo method is a good numerical method to approximate the price and the Greeks. Furthermore, there are a lot of general variance reduction techniques designed to improve the speed of the convergence of the method. One of them is the antithetic variates.

Table 5 shows that the Monte Carlo approximations with antithetic variates also converge to the exact solution with an increasing number of simulated paths as we have discussed. Table 5, in comparison with Table 2, shows that the results of the Monte Carlo method with antithetic variates are closer to the analytical solutions than the ordinary Monte Carlo method. Figure 2 shows that the Monte Carlo error of the Greeks estimator using antithetic variates is always smaller than the standard Monte Carlo method. We are able to see that the use of antithetic variables reduces variance, which implies reduces the Monte Carlo error. However, the reduction of the error we have seen in Table 5 is not as efficient as we might expect. Moreover, in Figure 2, we can see that for larger values of M the standard Monte Carlo error tends to be fairly close to the error of the Monte Carlo method using antithetic variates. We must also remember that it takes more than double the time to implement this technique.

This insignificant improvement may be due to the symmetric part of f. Suppose that $Y = f(\boldsymbol{Z})$ with $\boldsymbol{Z} = (Z1, \ldots, Zd) \sim N(0, I)$. Define the symmetric and antisymmetric parts of f, respectively, by $f_0(z) = \frac{f(z)+f(-z)}{2}$ and $f_1(z) = \frac{f(z)-f(-z)}{2}$. Glasserman (2004, p. 208) points out that antithetic sampling eliminates all variance if f is antisymmetric ($f = f_1$) and it eliminates no variance if f is symmetric ($f = f_0$). Some other alternatives to get variance reductions in financial engineering problems are stratified sampling (see e.g. Rubinstein and Kroese, 2008) and quasi Monte Carlo sampling (see e.g. Glasserman, 2004).

Table 5. Results of the Monte Carlo (MC) method with Antithetic Variates (AV) of price, delta, theta, and rho of European call option under the parameter setting: $N = 100$ and $\alpha = 2$. The exact values are $\Pi = 10.4505, \Delta = 0.6368, \Theta = -6.4140$ and $\rho = 53.2324$.

M	Π_M	$\hat{\Delta}_M$	$\hat{\Theta}_M$	$\hat{\rho}_M$
10^2	10.7895 ± 2.0783	0.6216 ± 0.0324	-6.6026 ± 0.5001	50.4285 ± 1.9255
10^3	10.7305 ± 0.6496	0.6311 ± 0.0101	-6.5748 ± 0.1565	50.5170 ± 0.6160
10^4	10.4422 ± 0.2035	0.6377 ± 0.0031	-6.4296 ± 0.0491	50.4591 ± 0.1911
10^5	10.4697 ± 0.0645	0.6372 ± 0.0010	-6.4447 ± 0.0155	50.4719 ± 0.0604
10^6	10.4508 ± 0.0204	0.6366 ± 0.0003	-6.43262 ± 0.0049	50.4768 ± 0.0191

3.5 An Option with Discontinuous Payoff

A binary option is an option with discontinuous payoff (see e.g. Hull, 2012). One type of binary option is cash-or-nothing option. It behaves similarly to a plain vanilla European option, but the payout is based on whether the option is on the money, not by how much it is in the money. Unlike with plain vanilla options, the payoff of binary options are fixed at the writing of the contract, not based on the price on the expiration date. The binary call option and binary put option has a payoff in equation (21) and (22) respectively.

$$f(S_T) = \begin{cases} 0 & \text{if } S_T \le K, \\ 1 & \text{if } S_T > K, \end{cases} \tag{21}$$

$$f(S_T) = \begin{cases} 1 & \text{if } S_T < K, \\ 0 & \text{if } S_T \ge K. \end{cases} \tag{22}$$

In the Black-Scholes model, the price at time t of a European binary call option with strike price K and maturity time T (see e.g. Hull (2012)):

$$E_Q[f(S_T)|\mathcal{F}_t] = e^{-r(T-t)}N(d_2(t,S_t)); \qquad (23)$$

and the price sensitivities:

$$\begin{aligned}
\Delta &= \frac{e^{-r(T-t)}N'(d_2)}{S\sigma\sqrt{T-t}};\\
\Theta &= re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}}\right);\\
\rho &= -(T-t)e^{-rT}N(d_2) + \frac{\sqrt{T-t}}{\sigma}e^{-rT}N'(d_2).
\end{aligned}$$

The price of a binary put option:

$$E_Q[f(S_T)|\mathcal{F}_t] = e^{-r(T-t)}N(-d_2(t,S_t)), \qquad (24)$$

and the price sensitivities:

$$\begin{aligned}
\Delta &= -\frac{e^{-r(T-t)}N'(d_2)}{S\sigma\sqrt{T-t}};\\
\Theta &= re^{-r(T-t)}(1-N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}}\right);\\
\rho &= -(T-t)e^{-rT}(1-N(d_2)) + \frac{\sqrt{T-t}}{\sigma}e^{-rT}N'(d_2),
\end{aligned}$$

where

$$d_1(t,x) = \frac{ln(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t,x) = \frac{ln(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and where N is the distribution function of an $N(0,1)$.

Comparing to the exact solution in Table 7, the method gives a quite good approximation of the results (see Table 6). We are able to see an increase of the error for the estimator of the Greeks in Table 6 as the current time becomes closer to maturity time. We may conclude that the methods also work quite well in the case of an option with discontinuous payoff for relatively large time to maturity.

Glynn (1989) pointed out that approximating the derivative function by a finite difference method for a discontinuous payoff can produce unpleasant errors. There are solutions to this problem, for instance, we can avoid differentiating the payoff by using the likelihood ratio method (see e.g. Broadie and Glasserman, 1996) or by using integration by parts to smooth the function (see e.g. Fourni´e et al, 1999).

Table 6. Results of simulation of binary call option under the parameter setting: $S = 2, K = 2, r = 0.1, \sigma = 0.2, T = 3, N = 40, M = 10^6$, and $\alpha = 2$.

t	Π_M	$\hat{\Delta}_M$	$\hat{\Theta}_M$	$\hat{\rho}_M$
0.0	0.5601 ± 0.0006	0.3446 ± 0.0007	0.0290 ± 0.0003	0.1306 ± 0.0000
1.5	0.5919 ± 0.0007	0.5976 ± 0.0008	0.0090 ± 0.0003	0.5606 ± 0.0005
2.0	0.5931 ± 0.0008	0.7849 ± 0.0008	-0.0084 ± 0.0003	0.7851 ± 0.0008
2.5	0.5812 ± 0.0009	1.1884 ± 0.0009	-0.0471 ± 0.0004	0.9029 ± 0.0009
2.75	0.5648 ± 0.0009	1.7409 ± 0.0009	-0.1065 ± 0.0004	0.7900 ± 0.0008
2.9	0.5445 ± 0.0009	2.8168 ± 0.0009	-2.7981 ± 0.0009	0.5644 ± 0.0006

Table 7. The price and Greeks of Black-Scholes (BS) binary call option under the parameter setting of Table 6.

t	Π_{bs}	Δ_{bs}	Θ_{bs}	ρ_{bs}
0.0	0.5599	0.3355	0.0291	0.3336
1.5	0.5920	0.6216	0.0094	0.8407
2.0	0.5930	0.8330	-0.0073	0.8785
2.5	0.5815	1.2890	-0.0449	0.7774
2.75	0.5649	1.9069	-0.0960	0.6169
2.9	0.5448	3.0976	-0.1933	0.4228

4. Conclusions

Computing Greeks is essential due to its ability to quantify the different aspects of risk associated with financial derivatives. There are a number of texts, e.g. Kloden and Platen (1992), Bjork (2003), and Glasserman (2004) which provide good materials of theories and methods that have been successfully applied to solve problems in mathematical finance.

For the test purposes, we presented numerical results using the Black-Scholes model which has exact solutions for the price and Greeks of the European options. In Monte-Carlo simulation, solutions are estimated by repeated simulations. This method produces the output statistics which have a confidence interval including the exact solution with a given probability. The Monte Carlo error can be a good measurement to determine when to end the simulation. As noted before, the size of a time-step does not affect the performance as much as the number of paths does. The possibility of estimating the Monte Carlo error, which is the dominant error in most of its computations, is the favourable feature of the Monte Carlo method. It is apparent that we need a very high path resolution in order to correctly price or compute Greeks of options. This makes the method computationally demanding.

We considered approximation of Greeks by finite difference method, which is easy to understand and implement. Besides, we chose the central difference approach instead of the forward difference method for its better convergence rate. We conclude that the finite difference method is a good method for estimating the Greeks using Monte Carlo simulation. It can also be applied to an option with discontinuous payoff. We have seen that, for a European binary call option, the method was able to approximate the correct analytic values of both the price and Greeks of the option. However, we discovered that this method can be problematic when the payoff function is discontinuous and the current time is close to maturity time. In addition, we have also seen that the antithetic variate technique reduces the Monte Carlo error but it does not produce a highly significant improvement, and takes more than the double time.

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